$$\left\{\frac{\quad}{(\quad)(\quad)}+\frac{(\quad)}{(\quad)}-1\right\}=\left\{\quad\right\}+\left\{\quad\right\}+\left\{\quad\right\}$$

$$=\quad$$

$$

$$(1+q)\left\{\frac{q}{(1-aq)(1-\frac{q}{a})}+\frac{q^2(1+q)(1+q^2)}{(1-aq)(1-aq^2)(1-\frac{q}{a})(1-\frac{q^2}{a})}+\right.$$
$$\left.+\frac{q^3(1+q)\cdots(1+q^4)}{\cdots}+\right\}$$
$$=(1+q+q^3+)\frac{1-aq+a^2q^3-\cdots}{1-(a+\frac{1}{a})q+(a^2+\frac{1}{a^2}+1)q^4-\cdots}$$
$$-(1-aq^2+a^2q^6-\cdots)$$

$$(1-q)(1-q^3)\cdots\left\{1-\frac{q(1-q)}{(1+q)(1+q^2)}+\frac{q^4(1-q)(1-q^2)}{(1+q)\cdots(1+q^4)}-\cdots\right.$$
$$=1-2\left(\frac{q}{1+q}+\frac{q^4}{1+q^2}+\frac{q^{12}}{1+q^3}+\frac{q^{20}}{1+q^4}+\cdots\right)$$
$$+2\left(\frac{q^7}{1-q+q^2}+\frac{q^{26}}{1-q^2+q^4}+\frac{q^{57}}{1-q^3+q^6}+\cdots\right)$$

If $f_a(q)=1+\frac{q}{1+aq+q^2}+\frac{q^4}{(1+aq+q^2)(1+aq^2+q^4)}+\cdots$
and $a^2+b^2=4$ then
$$\frac{b}{4}f_a(-q)+\frac{b+a+2}{4}f_{-a}(-q)-\frac{b}{2}f_b(q)$$
$$=\frac{(1-q^4)(1-q^8)\cdots}{(1+q)(1+q^3)\cdots}\cdot\frac{(1-(a+b)q+q^2)(1+q^2+q^4)\cdots}{(1+(a^2b^2-2)q^4+q^8)\cdots(1+(a^2b^2-2)q^8+q^{16})\cdots}$$

If $a^2+ab+b^2=3$ then
$$(a+1)f_{-a}(q)+(b+1)f_{-b}(q)-(a+b-1)f_{a+b}(q)$$
$$=\frac{\{(1-q^3)(1-q^6)(1-q^9)\cdots\}^2}{(1-q)(1-q^2)(1-q^3)\cdots}\Big/\frac{(1+ab(a+b)q^3+q^6)(1+ab(a+b)q^9+q^{18})\cdots}{}$$

$$\phi(\lambda) = \frac{(\lambda-1)}{\lambda} + \frac{}{(\lambda+1)(\lambda+1)\lambda^2} + \frac{}{\lambda}$$

$$= \frac{\lambda-1}{1} + \frac{}{(\lambda+1)\lambda} + \frac{}{(\lambda+1)(\lambda+1)\lambda^2} +$$

$$= \phi(\lambda)\lambda^{-1}z + \frac{}{\lambda^3+\lambda^2+\lambda+1} + \frac{2\lambda z+\lambda z-1}{}$$

$$= 1 + \frac{}{(\lambda-1)\lambda} + \frac{1}{} - 1$$

$$= 1 + \frac{}{(\lambda-1)\lambda^2} + \frac{}{(\lambda-1)\lambda^2} - 1$$

$$= \frac{}{(\lambda-1)\lambda^2} + \frac{}{(\lambda-1)\lambda^2} + \frac{}{(\lambda-1)\lambda^2} - 1$$

$$= \frac{}{(\lambda-1)\lambda^2} +

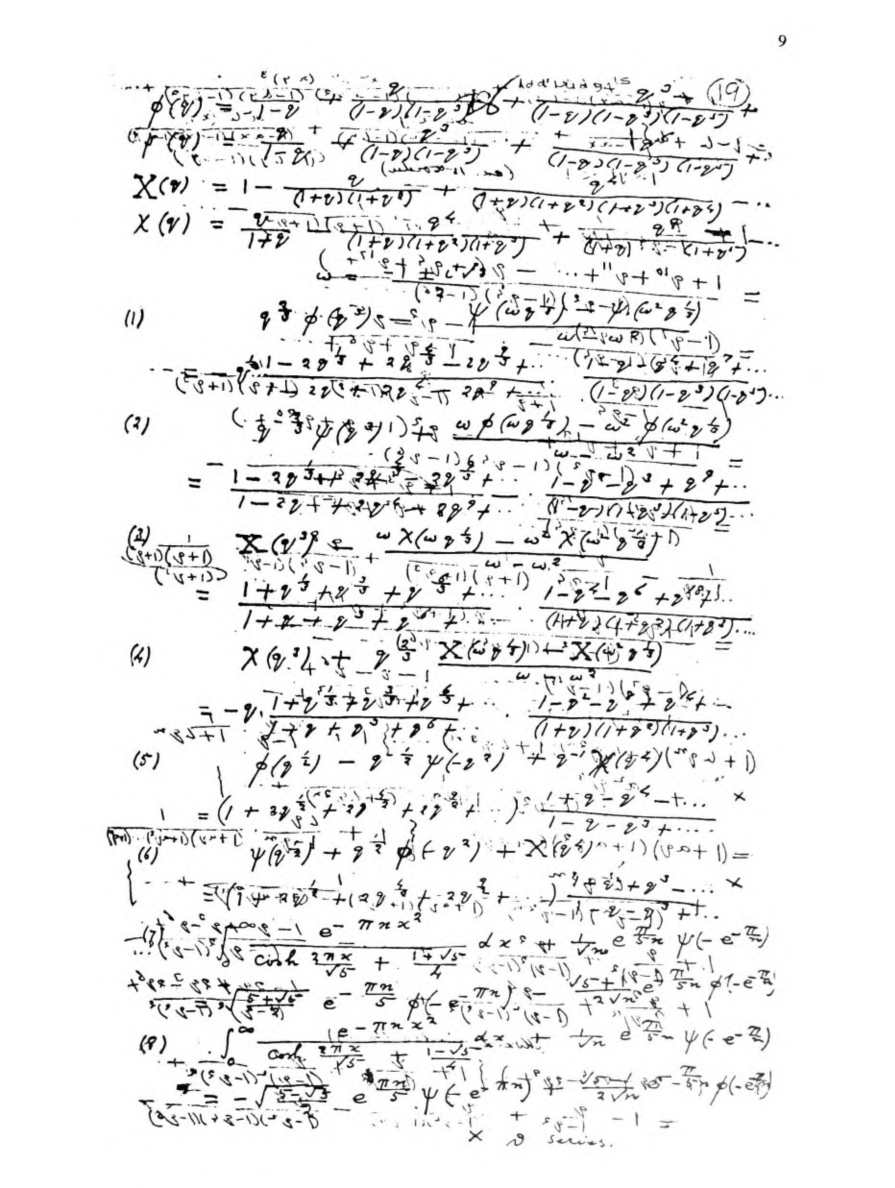
Generalizations simple and singular.

$$\frac{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}$$

$$\left\{\frac{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}\right\}$$

$$\left\{\frac{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}\right\}$$

$$\left\{\frac{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}{\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{\lambda+1}{\lambda}\right)\cdots\left(\frac{\lambda+1}{\lambda}\right)(\lambda+1)}\right\}$$



$$
\begin{aligned}
&= 1 + \frac{a_2}{1-a_2} + \frac{a_3}{(1-a_2)(1-a_3)} + \frac{a_4}{(1-a_2)(1-a_3)(1-a_4)} + \cdots \\
&= \frac{1 - a_2 a_3 a_4 \cdots}{(1-a_2)(1-a_3)(1-a_4)\cdots} \\
&= \frac{(1-a_2 a_3)(1-a_3 a_4)\cdots - a_2 a_3 a_4 \cdots}{(1-a_2)(1-a_3)(1-a_4)\cdots} \\
&= \frac{1 - a_2 a_3 + a_3 a_4 - a_2 a_3^2 a_4 + \cdots}{(1-a_2)(1-a_3)(1-a_4)\cdots}
\end{aligned}
$$

Cor:

$$
\begin{aligned}
&\text{I} \quad \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&= \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&\quad - \frac{\lambda-1}{2\lambda-1}\lambda \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&= 1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\text{II} \quad \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \left\{ 1 + \frac{\lambda-1}{3\lambda-1}\lambda + \frac{\lambda-1}{3\lambda-1}\lambda^2 + \cdots \right\} \\
&= \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&\quad + \frac{\lambda-1}{3\lambda-1}\lambda \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&\quad + \frac{\lambda-1}{3\lambda-1}\lambda^2 \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{\lambda-1}{2\lambda-1}\lambda^2 + \cdots \right\} \\
&= 1 + \frac{2\lambda-2}{2\lambda-1}\lambda + \frac{3\lambda-3}{3\lambda-1}\lambda + \frac{(2\lambda-2)(3\lambda-3)}{(2\lambda-1)(3\lambda-2)}\lambda^2 + \cdots \\
&= 1 + \frac{\lambda-1}{1-2\lambda}\lambda + \frac{\lambda-1}{1-3\lambda}\lambda + \frac{(\lambda-1)^2}{(1-2\lambda)(1-3\lambda)}\lambda^2 + \cdots \\
&= 1 + \frac{\lambda-1}{1-2\lambda}\lambda + \frac{\lambda-1}{1-3\lambda}\lambda + \frac{(\lambda-1)^2}{(1-2\lambda)(1-3\lambda)}\lambda^2 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&\text{III} \quad \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 + \cdots \right\} \\
&= \left\{ 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 + \cdots \right\} \\
&\quad - \frac{\lambda-1}{2\lambda-1}\lambda \left\{ 1 + \frac{\lambda-1}{3\lambda-1}\lambda + \frac{(\lambda-1)^2}{(3\lambda-1)(4\lambda-1)}\lambda^2 + \cdots \right\} \\
&\quad - \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 \left\{ 1 + \frac{\lambda-1}{4\lambda-1}\lambda + \frac{(\lambda-1)^2}{(4\lambda-1)(5\lambda-1)}\lambda^2 + \cdots \right\} \\
&= 1 + \frac{\lambda-1}{2\lambda-1}\lambda + \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 + \cdots \\
&\quad - \frac{\lambda-1}{2\lambda-1}\lambda - \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 - \cdots \\
&\quad - \frac{(\lambda-1)^2}{(2\lambda-1)(3\lambda-1)}\lambda^2 - \cdots \\
&= 1
\end{aligned}
$$

$$\frac{\lambda+1}{1}-\frac{(\lambda+1)(\lambda+1)}{(\lambda-1)\lambda}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^2\lambda^2}-\frac{(\lambda+1)(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^3\lambda^3}+\cdots$$

$$=1-\lambda^{-1}+\lambda^{-2}-\lambda^{-3}+\cdots$$

$$\frac{\lambda+1}{1}+\frac{(\lambda+1)(\lambda+1)}{(\lambda-1)\lambda}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^2\lambda^2}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^3\lambda^3}+\cdots$$

$$=1+\lambda^{-1}+\lambda^{-2}+\lambda^{-3}+\cdots$$

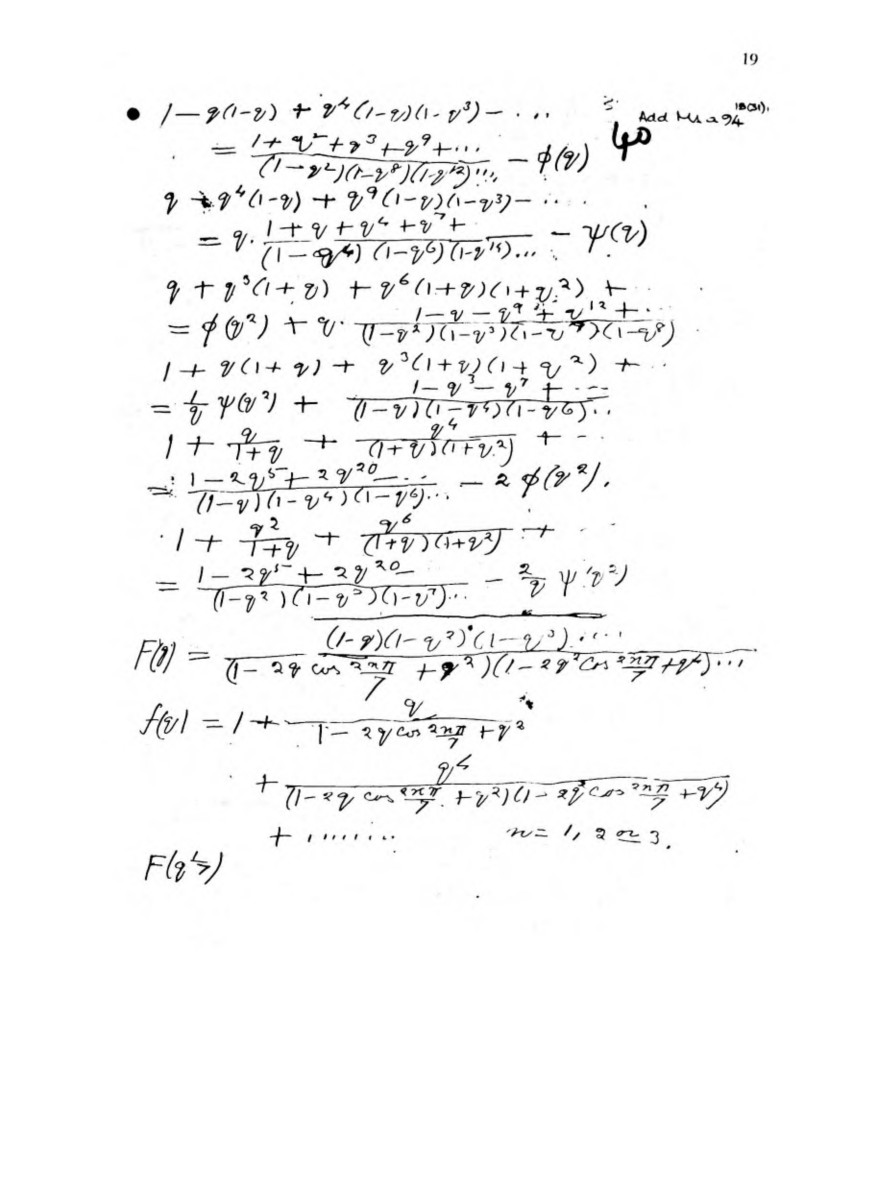
$$\frac{\lambda+1}{1}+\frac{(\lambda+1)(\lambda+1)}{(\lambda-1)\lambda}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^2\lambda^2}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^3\lambda^3}+\cdots$$

$$=1+2\lambda^{-1}+2\lambda^{-2}+2\lambda^{-3}+\cdots$$

$$\frac{\lambda+1}{1}+\frac{(\lambda+1)(\lambda+1)}{(\lambda-1)\lambda}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^2\lambda^2}+\frac{(\lambda+1)(\lambda+1)(\lambda+1)(\lambda+1)}{(\lambda-1)^3\lambda^3}+\cdots$$

$$=1+2\lambda^{-1}+3\lambda^{-2}+4

$$\frac{2}{3}\left\{\frac{q^{\frac{1}{2}}}{(1-q+q^2)} + \frac{q^{\frac{3}{2}}}{(1-q+q^2)(1-q^3+q^6)} + \cdots\right\}$$
$$+\frac{1}{3}\left\{\frac{q}{(1+q)^2} + \frac{q^2}{(1+q)^2(1+q^3)^2} + \cdots\right\}$$
$$= q^{\frac{1}{2}}\cdot\frac{(1-q^3-q^9+\cdots)^2}{(1-q^2)(1-q^4)(1-q^6)\cdots}$$
$$1 + q(1+q)(1+aq) + q^2(1+q)(1+aq^2)\cdots$$
$$= (1+q)(1+q^2)\cdots(1+aq)(1+aq^2)\cdots$$
$$\times\left\{\frac{1}{1+aq} + \frac{q^2(1-q)}{(1+aq)(1+aq^2)(1+aq^3)} + \frac{q^5(1-q)(1-q^3)}{(1+q^2)\cdots(1+q^3)\cdots}\right\}$$
$$2\left\{1 + q^2(1+q)(1+aq) + q^5(1+q)(1+q^2)(1+aq)(1+aq^2)(1+aq^3)+\cdots\right\}$$
$$+\frac{1}{1+a}\left\{1 + \frac{a}{(1+q)(1+aq)} + \frac{a^2}{(1+q)(1+q^2)(1+aq)(1+aq^2)}+\cdots\right\}$$
$$= \frac{1}{(1-q^2)(1-q^4)(1-q^6)\cdots}\cdot\left\{\frac{1}{1+a} + \left(\frac{q}{1+aq} + \frac{aq}{1+aq}\right) + \left(\frac{q^4}{1+aq^2} + \frac{q^3}{1+aq^2} + \frac{aq^2}{1+aq^2}\right)\right.$$
$$\left. + \frac{q^6}{1+aq^3} + \cdots\right) + \left(\frac{q^5}{a+q} + \frac{q^4}{a+q} + \frac{q^3}{a+q} + \frac{q^2}{a+q}\right) + \cdots\Big\}$$
$$\frac{1}{1+a}\left\{1 + \frac{a}{(a+1)(a+q)} + \frac{a^2}{(a+1)(a+q)(a+q^2)} + \cdots\right\}$$
$$+\frac{1}{(1+a)(1+q)}\left\{1 + a + a^2(1+a)(1+q) + a^3(1+a)(1+q)(1+aq)\right.$$
$$\left. + a^4(1+a)(1+q)(1+aq)(1+aq^2)+\cdots\right\}$$
$$= \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}\cdot\frac{1}{(1+a)(1+aq)(1+aq^2)\cdots}$$
$$\times\left\{\frac{1}{1+a} + \frac{aq}{1+a} + \frac{aq^3}{1+aq^2} + \frac{aq^3}{1+aq^3} + \cdots\right.$$
$$\left. + \frac{q}{a+q} + \frac{q^2}{a+q} + \frac{q^3}{a+q} + \cdots\right\}$$
$$(1+q)\left\{\frac{1}{1+a} + \frac{a(1-aq)(1-q)}{(1+a)(1+q)(1+aq)(1+aq^2)}\cdot\right.$$
$$\left. + \frac{a^2(1-aq)(1-aq^2)(1-q)(1-q^2)}{(1+a)(1+q)(1+aq)(1+aq^2)(1+aq^3)(1+q^2)}+\cdots\right\}$$
$$= (1+q) - (a+q)q^2 + (a^2+q)q^4 - \cdots$$
$$\frac{1}{1+a}\left\{1 + \frac{a}{(1+aq)(1+q)} + \frac{a^2}{(1+aq)(1+aq^2)(1+q)(1+q^2)}+\cdots\right\}$$
$$\times(1-q)(1-q^2)(1-q^3)\cdots$$
$$= \frac{1}{1+a} - \frac{q^2}{1+aq} + \frac{q^7}{1+aq^2} - \frac{q^{15}}{1+aq^3} + \cdots$$
$$- \frac{q^2}{a+q} + \frac{q^7}{a+q^2} - \frac{q^{15}}{a+q^3} + \cdots$$



$$F(q) = \frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1-2q\cos\frac{2\pi}{5}+q^2)(1-2q^2\cos\frac{4\pi}{5}+q^4)\cdots}$$

$$f(q) = 1 + \frac{q}{1-2q\cos\frac{2\pi}{5}+q^2} + \frac{q^4}{(1-2q\cos\frac{2\pi}{5}+q^2)(1-2q^2\cos\frac{4\pi}{5}+q^4)} + \cdots\cdots, \quad n = 1 \text{ or } 2.$$

$$F(q^{\frac{1}{5}}) = A(q) - 4q^{\frac{1}{5}}\cos^2\frac{2n\pi}{5}\cdot B(q) + 2q^{\frac{2}{5}}\cos\frac{4n\pi}{5}\cdot C(q) - 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}\, D(q);$$

$$f(q^{\frac{1}{5}}) = \left\{A(q) - 4\sin^2\frac{n\pi}{5}\,\phi(q)\right\} + q^{\frac{1}{5}}B(q) + 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}\,C(q) - 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}\left\{D(q) + 4\sin^2\frac{2n\pi}{5}\,\psi(q)\right\}$$

$$A(q) = \frac{1-q^2-q^3+q^9+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$B(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots}$$

$$C(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots}$$

$$D(q) = \frac{1+q^2+q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots}$$

$$\phi(q) = -1 + \left\{\frac{1}{1-q} + \frac{q^2}{(1-q)(1-q^2)(1-q^3)} + \frac{q^{20}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} + \cdots\right\}.$$

$$\psi(q) = -1 + \left\{\frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^4)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)} + \cdots\right\},$$

$$\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \cdots = 3\phi(q) + 1 - A(q).$$

$$\frac{q}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \cdots = 3\psi(q) + q\,D(q).$$

$$\frac{q}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \frac{q^{18}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \cdots = \phi(q) - q\cdot\frac{1+q^5+q^{14}+\cdots}{(1-q)(1-q^4)(1-q^6)\cdots}$$

$$\frac{q}{1-q} + \frac{q^{13}}{(1-q)(1-q^2)(1-q^3)} + \frac{q^{33}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} + \cdots = \psi(q) + q\cdot\frac{1+q^{19}+q^{52}+\cdots}{(1-q^2)(1-q^4)(1-q^6)\cdots}.$$

$$\left\{\cdots+\frac{z(z-1)_2(z-1)}{c_2 z}+\frac{z(z-1)}{a}+1\right\}\left\{\cdots(9z-1)(5z-1)(z-1)\right\}=$$

$$\cdots-(5z-1)(z-1)9z+(z-1)_2 z-1+$$

$$\left\{\cdots+\frac{(z-1)(z-1)}{9z}+\frac{(z-1)(z-1)}{9z}+\frac{z-1}{b}\right.$$

$$\cdots+(z+1)(z+1)_5 z+(z+1)5z+b$$

$$\cdots(z-1)(z-1)(z-1)\cdot\lambda=$$

$$\cdots+c_2 z^2+9_2 z+z+1$$

$$\left\{\cdots+\frac{(z-1)(z-1)}{9z}+\frac{z-1}{5z}\right\}-$$

$$\cdots+(z+1)(z+1)9z+(z+1)5z+b$$

$$=-\frac{\cdots(5z-1)(5z-1)(z-1)}{\cdots c_2 z^2-5z+z-1}$$

$$\left\{\cdots+\frac{(z-1)(z-1)}{9z}+\frac{z-1}{5z}\right\}z-\cdots+\frac{(z+1)(z+1)}{9z}+\frac{z+1}{z}+1$$

$$=\frac{\cdots(9z-1)(5z-1)(z-1)}{+c_2 z^2+9z+z+1}$$

$$\left\{-(9z-1)(5z-1)c_2 z+(z-1)_2 z-1\right\}+$$

$$\cdots+(c_2 z+1)(z+1)9z+(z+1)5z+b$$

$$=\frac{\cdots(5z-1)(5z-1)(z-1)}{+c_2 z^2+9z+z+1}$$

$$\left\{+\frac{(9z-1)(5z-1)}{9z}+\frac{z-1}{5z}\right\}z+\cdots+\frac{(z+1)(z+1)}{5z}+\frac{z+1}{z}+1=$$

$$=\frac{\cdots(9z-1)(5z-1)(z-1)}{+c_2 z^2+9z+z+1}$$

$$\left\{\cdots+\frac{(z-1)(5z-1)}{9z}+\frac{z-1}{5z}\right\}-\cdots+\frac{(z+1)(z+1)}{5z}+\frac{z+1}{z}+1$$

$$=-\frac{\cdots(5z-1)(5z-1)(z-1)}{\cdots c_2 z^2-5z+z-1}$$

$$\left\{\cdots+\frac{(z-1)(5z-1)}{9z}+\frac{z-1}{5z}\right\}z-\cdots+\frac{(z+1)(z+1)}{9z}+\frac{z+1}{z}+1$$

$$=\frac{\cdots(9z-1)(5z-1)(z-1)}{+c_2 z^2+9z+z+1}$$

$$\left\{\cdots+\frac{(z-1)(z-1)}{c_2 z}+\frac{z-1}{a}\right\}-\cdots+(z+1)(z+1)c_2 z+(z+1)z+1$$

$$=\frac{\cdots(z_2 z-1)(5z-1)(z-1)}{+c_2 z^2-5z+z-1}$$

$$\left\{\cdots+(c_2-1)z_2-z\right\}z-\cdots+\frac{(z+1)(z+1)}{9z}+\frac{z+1}{z}+1$$

$$=\frac{\cdots(z+1)(z+1)z_2+(z+1)z+1}{+c_2 z^2+9z+z+1}$$

$$\left\{\cdots+\frac{(z-1)(z-1)}{c_2 z}+\frac{z-1}{a}\right\}z+\cdots+\frac{(z+1)(z+1)}{9z}+\frac{z+1}{z}+1=$$

$$\left\{\cdots+(9z-1)(5z-1)(z-1)_2 z\right\}\left\{\cdots+\frac{z(z-1)}{a}+1\right\}=$$

$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{\lambda^3+\lambda^2+\lambda+1} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{(\lambda+1)(\lambda^2+1)} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

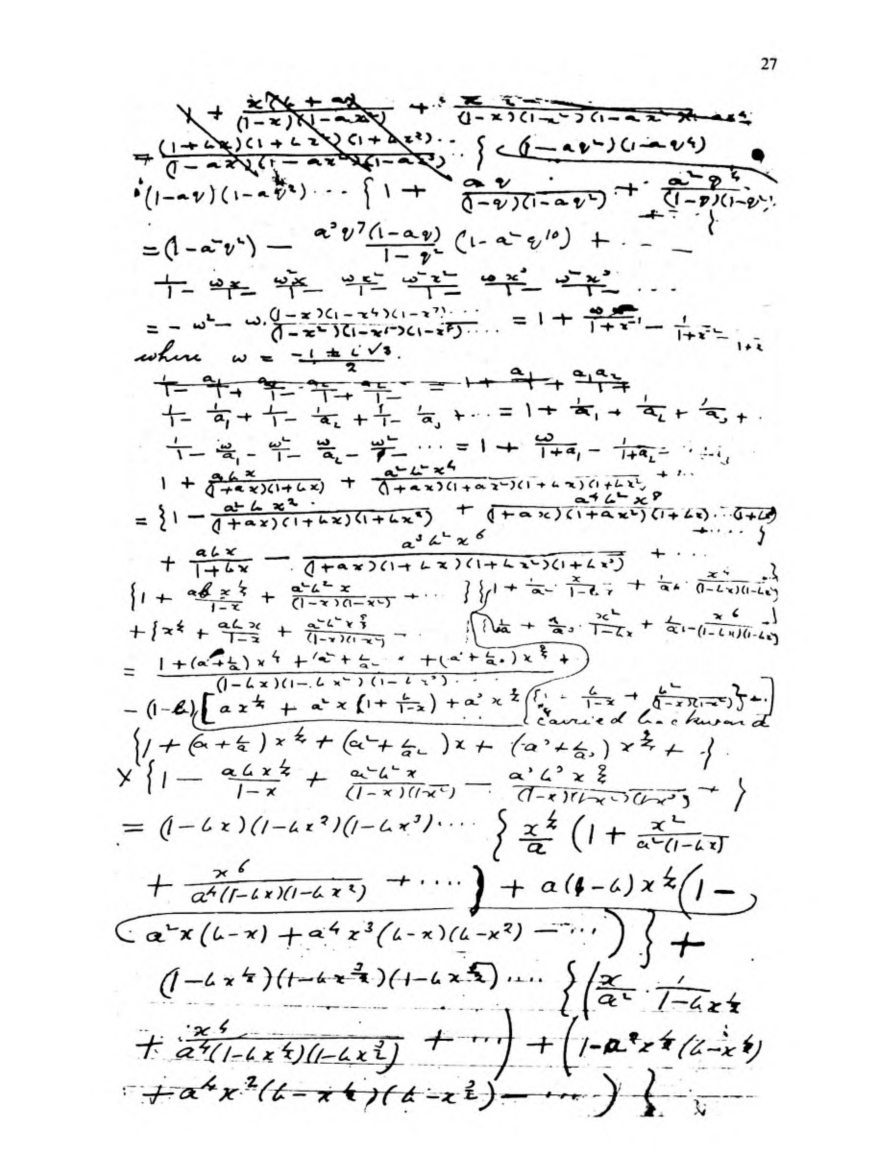
$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{(\lambda+1)(\lambda^2+1)} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{(\lambda+1)(\lambda^2+1)} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{(\lambda+1)(\lambda^2+1)} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

$$\frac{(\lambda-1)(\lambda-1)(\lambda-1)}{(\lambda+1)(\lambda^2+1)} = \frac{\lambda^3-3\lambda^2+3\lambda-1}{\lambda^3+\lambda^2+\lambda+1} = 1 - \frac{6\lambda^2}{\lambda^3+\lambda^2+\lambda+1}$$

$$\left[\begin{array}{c} \\ \end{array}\right.$$



$$\phi(a,x) = 1 + \frac{x}{1-x} + \frac{a^2x^2}{(1-x)(1-x^2)} + \cdots$$

And Ramanujan (37) (38)

$$\phi(a,x)\,\phi\left(b,\frac{1}{x}\right) = 1 + \frac{x}{1-x}(a-b) + \frac{x^2}{(1-x)(1-x^2)}(a-b)(a-bx) + \frac{x^3}{(1-x)(1-x^2)(1-x^3)}(a-b)(a-bx)(a-bx^2) + \cdots$$

If $\phi(a,x,y) = 1 + \frac{ax}{1-xy} + \frac{a^2x^2}{(1-xy)(1-x^2y^2)} + \cdots$

then $\phi(a,x,y)\,\phi(b,x,y) = 1 + \frac{ax+by}{1-xy} + \frac{(ax+by)(ax^2+by^2)}{(1-xy)(1-x^2y^2)} + \frac{(ax+by)(ax^2+by^2)(ax^3+by^3)}{(1-xy)(1-x^2y^2)(1-x^3y^3)} + \cdots$

(i) $\frac{1}{1+av} - \frac{a^2v^2}{(1+av)(1+av^2)} + \frac{a^4v^6}{(1+av)(1+av^2)(1+av^3)} - \cdots$

$= (1-av) + a^2v^2\cdot\frac{1+v}{1+av}(1-av^3) + a^4v^{12}\frac{(1+v)(1+v^2)}{(1+av)(1+av^2)}(1-av^5) + \cdots$

(ii) $1 - \frac{a^2v^2}{(1+av)(1+av^2)} + \frac{a^4v^6}{(1+av)(1+av^2)(1+av^3)(1+av^4)} - \cdots$

$= (1-a^2v^2) + a^2v^3\frac{1+v}{1+av^2}(1-a^2v^4) + a^4v^{11}\frac{(1+v)(1+v^2)}{(1+av^2)(1+av^4)}(1-a^2v^6) + \cdots$

(i) & (ii) deduced from each other.

$$1 + av + a^2v^4 + a^3v^9 + \cdots$$

$$= 1 + \frac{av}{1+av^2} + \frac{a^2v^3(1+v^3)}{(1+av^2)(1+av^4)} + \frac{a^3v^6(1+v)(1+v^3)}{(1+av^2)(1+av^4)(1+av^6)} + \cdots$$

$$= 1 + \frac{ax}{1+ax^2}\cdot\frac{1+4x}{1+4x} + \frac{a^2x^3}{(1+ax^2)(1+ax^4)}\cdot\frac{(1+4x)(1+4x^3)}{(1+4x)(1+4x^2)} + \cdots$$

$$= (1+ax) - a^2x^5\frac{(1-ax)(1-x)}{(1+ax^2)(1+4x^2)}(1+ax^6) + a^4x^{18}\frac{(1-ax)(1-ax^3)}{(1+ax^2)(1+ax^4)}\cdot\frac{(1-x)(1-x^3)}{(1+4x^2)(1+4x^4)}(1+ax^9) - \cdots$$

$$\frac{(1+ax)(1+ax^4)\cdots}{(1+4x)(1+4x^2)\cdots}\left\{1 + x\cdot\frac{1-ax}{1-x} + x^3\cdot\frac{(1-ax)(1-ax^2)}{(1-x)(1-x^2)} + \cdots\right\}$$

$$= (1+ax) + ax^3\cdot\frac{1-ax}{1-x}\cdot\frac{1+ax}{1+4x}(1+ax^3) + \cdots$$

i.e. $+ a^2x^{10}\frac{(1-ax)(1-ax^2)}{(1-x)(1-x^2)}\cdot\frac{(1+ax)(1+ax^2)}{(1+4x)(1+4x^2)}(1+ax^5) + \cdots$

$$(1-\lambda x)(1-\lambda x^2)(1-\lambda x^3)\cdots\left\{1 + \frac{\lambda x}{(1-x)(1-ax)} + \frac{\lambda^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)} + \cdots\right.$$

$$= 1 + \lambda x^2\cdot\frac{a-\lambda}{1-x}\cdot\frac{1-\lambda x^2}{1-ax} + \lambda^2x^8\frac{(a-\lambda)(a-\lambda x)}{(1-x)(1-x^2)}\cdot\frac{(1-\lambda x)(1-\lambda x^2)}{(1-ax)(1-ax^2)} + \cdots$$

$$1 - av + a^2v^3 - a^3v^6 + \cdots$$

$$= \frac{1}{1+av} - \frac{a^2v^2(1-v)}{(1+av)(1+av^2)(1+av^3)} + \frac{a^4v^6(1-v)(1-v^2)}{(1+av)\cdots(1+av^4)} - \cdots$$

$$\frac{1}{1+av} - \frac{v^2}{(1+av)(1+av^2)(1+av^3)}\cdot\frac{1-a^2v^2}{1-v^2} + \frac{v^6}{(1+av)\cdots(1+av^5)}\cdot\frac{(1-a^2v^2)(1-a^2v^4)}{(1-v^2)(1-v^4)} - \cdots$$

$$= \frac{(1-av^2)(1-av^4)(1-av^6)\cdots}{(1-a^2v^2)(1-a^2v^4)(1-a^2v^6)\cdots}\left\{(1-av) + \frac{1-av}{1-v}\cdot\frac{1-a^2v^3}{1-v^2}\cdot av^2(1-av^3) + a^2v^{10}\frac{(1-av)(1-av^2)}{(1-v)(1-v^2)}\cdot\frac{(1-a^2v^5)}{(1-v^4)(1-v^5)}\right.$$

$$\left.\times(1-av^5) + \cdots\right\}$$

$$(1+a)\left\{\frac{1}{a+cx} - \frac{bx}{(a+cx)(a+cx^2)} + \frac{b^2x^3}{(a+cx)(a+cx^2)(a+cx^3)} - \frac{(1-cx)(1-cx^2)}{(1+bx)(1+bx^2)} - \cdots\right\}$$

$$= 1 + \frac{x(1-cx)}{(1+ax)(1+bx)} + \frac{x^2(1-cx)(1-cx^2)}{(1+ax)(1+ax^2)(1+bx)(1+bx^2)} + \cdots$$

$$+ \frac{(1-cx)(1-cx^2)(1-cx^3)\cdots}{(1+ax)(1+ax^2)\cdots(1+bx)(1+bx^2)\cdots}\left\{\frac{1}{a+cx} - \frac{bx}{(a+cx)(a+cx^2)} + \frac{b^2x^3}{(a+cx)(a+cx^2)(a+cx^3)} - \cdots\right\}$$

$$\left(1+\frac{b}{a}\right)\left\{1 - \frac{b}{a}\cdot\frac{cx}{1+bx} + \left(\frac{b}{a}\right)^2\frac{cx^3}{(1+bx)(1+bx^2)} - \cdots\right\}$$

$$= \frac{1}{a}\left\{\frac{1}{1+bx} - \frac{b}{a}\cdot\frac{cx}{1+ax}\cdot\frac{1}{(1+bx)(1+bx^2)} + \left(\frac{b}{a}\right)^2\frac{cx^3}{(1+ax)(1+ax^2)}\cdot\frac{1}{(1+bx)(1+bx^2)(1+bx^3)} - \cdots\right\}$$

$$+ \frac{1}{x}\left\{x - \frac{b}{a}\cdot\frac{cx^3}{1+ax}\cdot\frac{1}{(1+bx)(1+bx^2)} + \left(\frac{b}{a}\right)^2\frac{cx^6}{(1+ax)(1+ax^2)}\cdot\frac{1}{(1+bx^2)(1+bx^3)(1+bx^4)} - \cdots\right\}$$

$$(1+ax)(1+ax^2)\cdots\left\{1 + \frac{bx}{1-x}\cdot\frac{1}{(1+ax)(1+ax^2)\cdots(1+ax^n)} + \frac{b^2x^3}{(1-x)(1-x^2)}\cdot\frac{1}{(1+ax)(1+ax^2)\cdots(1+ax^{n+2})} + \cdots\right\}$$

is symmetric with respect to a and b.

$$1 + \frac{x(a-c)}{(1-x)(1-ax)} + \frac{x^3(a-c)(a^2-c)}{(1-x)(1-x^2)(1-ax)(1-ax^2)} + \cdots$$

$$= \frac{(1-cx)(1-cx^2)(1-cx^3)\cdots}{(1-ax)(1-ax^2)(1-ax^3)\cdots}\left\{1 + \frac{x^2}{1-x^2}\cdot\frac{c+a^2x}{1-cx} + \frac{x^6}{(1-x^2)(1-x^4)}\cdot\frac{(c+a^2x)(c+a^2x^3)}{(1-cx)(1-cx^3)} + \cdots\right\}$$

$$\frac{ax}{1+a}\left\{1 - \frac{ax}{1+ax} + \frac{a^2x^2}{(1+ax)(1+ax^2)} - \cdots\right\}$$

$$\times (1+ax)(1+ax^2)(1+ax^3)\cdots$$

$$= \frac{ax}{1+a} + \frac{a^2x^2}{1+a}\cdot\frac{1}{(1+ax)(1+ax^2)} + \cdots$$

$$= \frac{ax}{1+a}\left\{1 + \frac{ax^2}{1+ax^2} + \cdots\right\}$$

$$= \frac{ax}{1+a}\left\{1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + \cdots\right\}$$

$$1 + \frac{x^2(1+ax)(1+\frac{x}{a})}{(1-x^2)(1-x^4)} + \frac{x^8(1+ax)(1+ax^3)(1+\frac{x^2}{a})(1+\frac{x^3}{a})}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)} + \cdots$$

$$= \frac{1 + x^3(a+\frac{1}{a}) + x^{12}(a^2+\frac{1}{a^2}) + \cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots}$$

e.g. $1 + \frac{x(1+x)}{(1-x)(1-x^2)} + \frac{x^4(1+x)(1+x^2)}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \cdots$

$$= \frac{1 + x^2 + x^4 + x^{10} + \cdots}{1 - x - x^3 + x^5 + \cdots}$$

$$1 + \frac{x^2(1-x)^2}{(1-x^2)(1-x^4)} + \frac{x^8(1-x)^2(1-x^2)^2}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)} + \cdots$$

$$= \frac{1 + x + x^2 + x^5 + x^7 + \cdots}{1 + x + x^2 + x^4 + \cdots}$$

$$1 + \frac{x^2(1-x^3)}{(1-x)(1-x^2)(1-x^4)} + \frac{x^8(1-x^3)(1-x^5)}{(1-x)(1-x^2)(1-x^4)(1-x^6)(1-x^8)} + \cdots$$

$$= \frac{1 - x - x^5 + x^8 + \cdots}{1 - x - x^2 + x^5 + \cdots} \quad (1-x^9)(1-x^{27})\cdots$$

$$1 + \frac{x}{1-x}\cdot\frac{(1+ax)(1+\frac{x}{a})}{1-x^4} + \frac{x^6}{(1-x)(1-x^3)}\cdot\frac{(1+ax)}{(1-x^4)(1-\frac{x^2}{a})(1+\frac{x^2}{a})} + \cdots$$

$$= \frac{1 + x^2(a+\frac{1}{a}) + x^8(a^2+\frac{1}{a^2}) + \cdots}{1 - x - x^3 + x^6 + \cdots}$$

$$1 \pm \frac{x(1+x)}{1-x^4} + \frac{x^5(1+x)(1+x^2)}{(1-x^4)(1-x^6)} \pm \cdots$$

$$= \frac{1 \pm (x + x^2) + (x^5 + x^7) \pm \cdots}{(1-x^4)(1-x^6)(1-x^{10})\cdots}$$

$$1 + \frac{x^2(1+x)}{1-x^4} + \frac{x^8(1+x)(1+x^3)}{(1-x^4)(1-x^6)} + \cdots = \frac{1 + x^2 + x^3 + x^7 + \cdots}{(1-x^4)(1-x^6)\cdots}$$

$$1 + \frac{x(1+x^3)}{(1-x^2)(1-x^4)} + \frac{x^6(1+x^3)(1+x^6)}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)} + \cdots$$

$$= \frac{1 - x^2 - x^6 + \cdots}{1 - x - x^2 + \cdots} \quad (1+x^6)(1+x^{10})\cdots$$

$$1 + \frac{x(1-x^3)}{(1-x)^2(1-x^4)} + \frac{x^6(1-x^3)(1-x^6)}{(1-x)^2(1-x^4)^2(1-x^6)} + \cdots$$

$$= \frac{1 + x^2 + x^6 + \cdots}{1 - x - x^2 + \cdots} \quad (1-x^6)(1-x^{10})\cdots$$

$$* \quad 1 - \frac{(1+ax)(1+\frac{x}{a})}{1-x^4} + \frac{(1+ax)(1+ax^2)(1+\frac{x^2}{a})(1+a)}{(1-x^4)^2(1-x^6)} - \cdots$$

$$= \frac{1 - x(a+\frac{1}{a}) + x^5(a^2+\frac{1}{a^2}) - \cdots}{x^2(1+x^2+x^6+\cdots)}$$

* Summable

$$(1-x^2)(1-x^{10})\cdots$$

$$= \frac{1}{1-x} - \frac{x^2}{1-x^2} - \frac{x^4}{(1-x)(1-x^6)} + \frac{x^9}{(1-x^2)(1-x^8)}$$

$$(1-x^{11})(1-x^{10})\cdots \quad (1+x^2+x^4+\cdots)$$

$$= \frac{1}{1-x} - \frac{x^2}{1-x^2} - \frac{x^4}{(1-x)(1-x^6)} + \frac{x^{11}}{\cdots} + \frac{x^9}{\cdots} - \cdots$$

$$\frac{x}{1+x^2} - \frac{x^4}{(1+x^2)(1+x^3)} + \frac{x^{11}}{(1+x^2)(1+x^3)(1+x^5)} - \cdots$$

$$= \tfrac{1}{2}(1-x^2+x^6-x^{12}+\cdots) - \tfrac{1}{2}\cdot\frac{1-2x+2x^4-2x^9}{(1-x^2)(1-x^4)(1-x^6)}\cdots$$

$$\frac{1}{1+x} - \frac{x^4}{(1+x)(1+x^4)} + \frac{x^7}{(1+x)(1+x^4)(1+x^7)} - \cdots$$

$$= \tfrac{1}{2}(1-x^2+x^6-x^{12}+\cdots) + \tfrac{1}{2}\cdot\frac{1-2x+2x^4}{(1-x^2)(1-x^6)}\cdots$$

$$1 - \frac{2x}{1+2x} + \frac{2^2x^3}{(1+2x)(1+2x^2)} - \cdots$$

$$= 1 - \frac{2x}{1+2x^2} + \frac{2^2x^4}{(1+2x^2)(1+2x^4)} - \cdots = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1-x} - \frac{x^3}{(1-x)(1-x^3)} + \frac{x^6}{(1-x)(1-x^3)(1-x^6)} - \cdots = 1 + x^2 - x^{10} - x^{16} + \cdots$$

$$1 + \frac{x^2}{1-x} + \frac{x^3}{(1-x)(1-x^3)} + \frac{x^6}{(1-x)(1-x^3)(1-x^6)} = \cdots$$

$$\frac{x}{1+x^2} - \frac{x^7}{(1+x^2)(1+x^5)} + \frac{x^{18}}{(1+x^2)(1+x^5)(1+x^{11})} - \cdots$$

$$= \tfrac{1}{2}(1-x^4+x^{10}-\cdots) + \tfrac{1}{2}x(1-x+x^{14}-\cdots) - \tfrac{1}{2}\cdot\frac{1-x^2-x^5+\cdots}{(1+x)(1+x^5)}\cdots$$

$$\frac{x}{1-x^2} - \frac{x^7}{(1-x^2)(1-x^5)} + \cdots$$

$$= \tfrac{1}{2}(1+x^4-x^{11}-x^{12}+\cdots) + \tfrac{1}{2}x(1+x-x^{12}\cdots) - \tfrac{1}{2}\cdot\frac{(1-x^2)(1-x^{10})\cdots}{1-x^{12}}$$

$$\frac{1}{1+x^2} - \frac{x^4}{(1-x^2)(1-x^7)} + \cdots = \tfrac{1}{2}(1+x^4-x^{10}-\cdots) + \tfrac{1}{2}x(1+x-x^{12}) + \tfrac{1}{2}\cdot\frac{(1-x^2)(1-x^{10})\cdots}{1-x^{12}}\cdots$$

$$\frac{1}{1+x^2} - \frac{x^4}{(1+x^2)(1+x^7)} + \cdots = \text{and so on.}$$

$$1 - \frac{a^2x}{1-a^2x^2} + \frac{a^4x^4}{(1-a^2x^2)(1-a^2x^4)} - \cdots$$

$$= 1 - a\left(\frac{ax}{1+ax} + \frac{a^2x^2}{(1+ax)(1+ax^2)} + \cdots\right)$$

$$= 1 - \frac{a^2x}{1+ax} + \frac{a^4x^3}{(1+ax)(1+ax^2)} - \frac{a^6x^6}{(1+ax)(1+ax^2)(1+ax^3)} + \cdots$$

$$1 - x + x^2 - x^3 + x^4 - \cdots$$

$$= 1 - \frac{x}{1+x} + \frac{x^3}{(1+x)(1+x^2)} - \frac{x^6}{(1+x)(1+x^2)(1+x^3)} + \cdots$$

$$= \frac{1}{1+x} + \frac{x^3}{(1+x)(1+x^2)(1+x^3)} + \frac{x^{10}}{(1+x)(1+x^2)\cdots(1+x^5)}$$

$$= 1 - \frac{x}{(1+x)(1+x^2)} - \frac{x^6}{(1+x)(1+x^2)(1+x^3)(1+x^5)} - \cdots$$

$$1 + \frac{x}{(1+x)(1+\frac{x}{a})} + \frac{x^2}{(1+ax)(1+ax^2)(1+\frac{x}{a})(1+\frac{x^2}{a})} + \cdots$$

$$= (1+a)(1 - a^2x + a^3x^3 - a^5x^5 + a^6x^7 - \cdots)$$

$$- a\,\frac{1 - a^2x + a^4x^3 - a^6x^6 + \cdots}{(1+ax)(1+\frac{x}{a})(1+ax^2)(1+\frac{x^2}{a})} - \cdots$$

$$1 + \frac{abx}{(1-ax)(1-bx)} + \frac{a^2b^2x^4}{(1-ax)(1-ax^2)(1-bx)(1-bx^2)} + \cdots$$

$$= 1 + a\left\{\frac{bx}{1-ax} + \frac{b^2x^2}{(1-ax)(1-ax^2)} + \frac{b^3x^3}{(1-ax)(1-ax^2)(1-ax^3)}\right\}$$

$$1 - ax + a^3x^2 - a^6x^8 + a^6x^{16} - \cdots$$

$$= \frac{1}{1+ax} - \frac{a^2x^2}{(1+ax)(1+ax^3)} + \frac{a^4x^6}{(1+ax)(1+ax^3)(1+ax^5)} - \cdots$$

$$\frac{x}{(1+ax)(1+\frac{x}{a})} + \frac{x^3}{(1+ax)(1+ax^3)(1+\frac{x}{a})(1+\frac{x^3}{a})} + \cdots$$

$$= (a - a^2x + a^4x^5 - a^5x^8 + \cdots$$

$$- \frac{a - a^3x^4 + a^5x^6 - a^7x^{12} + \cdots}{(1+ax)(1+\frac{x}{a})(1+ax^3)(1+\frac{x^3}{a})} - \cdots$$

$$1 + x^2 - x^{10} - x^{16} + \cdots$$

$$= \frac{1}{1+x} + \frac{x}{(1+x)(1+x^3)} + \frac{x^2}{(1+x)(1+x^3)(1+x^7)} + \cdots$$

$\psi(x) = 1 + ax + a^2x^3 + a^3x^6 + a^4x^{10} + \cdots$

(i) $\psi(x) + \omega\psi(x\omega) + \omega^2\psi(x\omega^2) = 0$

(ii) $\psi(x) + \omega^2\psi(x\omega) + \omega\psi(x\omega^2) = 3ax\psi(x^3)$

$$\frac{1}{1+ax} - \frac{a^2x^2}{(1+ax)(1+ax^2)} + \frac{a^4x^4}{(1+ax)(1+ax^2)(1+ax^3)} - \cdots$$

$$= \frac{1}{2}(1 - a^2x^3 + a^3x^6 - a^5x^{10} + \cdots) + \frac{1}{2}\{\psi(x) - 3ax\psi(x^3)\}$$

$$\frac{ax}{1+a^2x^2} - \frac{a^3x^3}{(1+ax^2)(1+ax^3)} + \frac{a^5x^{12}}{(1+ax^2)(1+ax^3)(1+ax^4)} - \cdots$$

$$= \frac{1}{2}(1 - a^2x^3 + a^3x^6 - a^5x^{10} + \cdots) - \frac{1}{2}\{\psi(x) - 3ax\psi(x^3)\}.$$

$$1 - a\left\{\frac{x}{(1+ax)(1+\frac{x^2}{a})} + \frac{x^{13}}{(1+ax)(1+\frac{x^2}{a})(1+ax^2)(1+\frac{x^3}{a})} + \frac{x^{37}}{\cdots} + \cdots\right\}$$

$$- \frac{1}{a}\left\{\frac{x}{(1+\frac{x}{a})(1+ax^2)} + \frac{x^{13}}{(1+\frac{x}{a})(1+ax^2)\cdots}\right\}$$

$$- \left\{\frac{x^4}{(1+ax^2)(1+\frac{x^2}{a})} + \frac{x^{14}}{(1+ax^2)(1+\frac{x^2}{a})(1+ax^3)(1+\frac{x^3}{a})} + \cdots\right\}$$

$$= \frac{\{(1-x^2)(1-x^4)(1-x^6)\cdots\}^3}{(1-x^6)(1-x^{12})(1-x^{18})\cdots} \cdot \frac{1}{1 + x(a+\frac{1}{a}) + x^4(a^2+\frac{1}{a^2}) + \cdots}$$

$$\left[1 - \frac{ax}{1+ax^2} + \frac{a^2x^4}{(1+ax^2)(1+ax^4)} - \cdots\right.$$

$$= 1 - \frac{ax}{1+ax} + \frac{a^2x^3}{(1+ax)(1+ax^2)} - \frac{a^3x^6}{\cdots}\left.\right]$$

$$\frac{1}{1+ax} + \frac{ax^2}{(1-x^4)(1+ax^3)} + \frac{a^2x^6}{(1-x^2)(1-x^4)(1+ax^3)}$$

$$= (1+ax^2)(1+ax^4)(1+ax^6)\cdots\left\{1 - \frac{ax}{1+ax} + \frac{a^2x^3}{(1+ax)\cdots} - \frac{a^3x^6}{(1+ax)(1+ax^2)(1+ax^3)} + \cdots\right\}$$

$$\frac{1+x+x^3+\cdots}{1-x^2-x^6+\cdots} = 1 + \frac{x}{1-x^2} + \frac{x^3}{(1-x^2)(1-x^4)} + \cdots = \frac{1-x^2-x^6+\cdots}{1-x-x^2+\cdots}$$

$$\frac{1+\cdots}{1} = 1 + \frac{x^2}{1-x^2} + \frac{x^6}{(1-x^2)(1-x^4)} + \cdots = \frac{1-x-x^2+\cdots}{1-x-x^2+\cdots}$$

$$1 + \frac{2x}{(1+x^2)} \qquad \frac{2x^{12}}{(1+x^3)(1+x^6)} \cdot x^7$$

$$= 4\left\{ \frac{x}{(1+x)(1+x^2)} + \frac{x^7}{(1+x)(1+x^2)(1+x^4)(1+x^8)} + \frac{x^{19}}{\cdots} \right\}$$

$$+ \frac{(1-2x+2x^4-2x^9+\cdots)^2}{(1-x^3)(1-x^6)(1-x^9)\cdots}$$

$$\frac{1}{1+ax} - \frac{a^2x^2}{(1+ax)(1+ax^2)} + \frac{a^4x^7}{(1+ax)(1+ax^2)(1+ax^3)}$$

$$+ \frac{ax}{1+ax^2} - \frac{a^3x^5}{(1+ax^2)(1+ax^3)} + \frac{a^5x^{12}}{(1+ax^2)(1+ax^3)(1+ax^4)}$$

$$= 1 - a^2x^3 + a^3x^6 - a^5x^{15} + a^6x^{21} -$$

$$1 + \frac{x^3}{(1+ax^3)(1+\frac{x^3}{a})} + \frac{x^{12}}{(1+ax^2)(1+\frac{x^2}{a})(1+ax^6)(1+\frac{x^6}{a})}$$

$$- (1+a)\left\{ \frac{x}{(1+ax)(1+\frac{x^2}{a})} + \frac{x^7}{(1+ax)(1+\frac{x^2}{a})(1+ax^4)(1+\frac{x^4}{a})} \right\}$$

$$- (1+\tfrac{1}{a})\left\{ \frac{x}{(1+\frac{x}{a})(1+ax^2)} + \frac{x^7}{(1+\frac{x}{a})(1+ax^4)} \cdots \right\}$$

$$= \left\{ \frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^3)(1-x^6)(1-x^9)\cdots} \right\}^2 \cdot \frac{1}{(1+ax)(1+ax^2)(1+ax^3)\cdots}$$

$$\times (1+\tfrac{x}{a})(1+\tfrac{x^2}{a})(1+\tfrac{x^3}{a})\cdots$$

$$\tfrac{1}{2}(1 - a^2x + a^2x^2 - a^4x^3 + \cdots) - \tfrac{1}{2}(1 - a^2x - a^2x^2 + \cdots)$$

$$= \frac{a^2x^2}{1+a^2x^3} - \frac{a^7x^8}{(1+a^2x^3)(1+a^2x^7)} + \frac{a^{11}x^{18}}{1+\cdots}$$

$$\tfrac{1}{2}(a+1)\left\{ 1 + \frac{x}{(1+ax)(1+\frac{x}{a})} + \frac{x^4}{(1+ax)(1+\frac{x}{a})(1+ax^2)(1+\frac{x^2}{a})} + \cdots \right\}$$

$$+ \frac{1}{2(a-1)}\left\{ 1 + \frac{x}{(1-ax)(1-\frac{x}{a})} + \frac{x^4}{(1-ax)(1-\frac{x}{a})(1-ax^2)(1-\frac{x^2}{a})} + \cdots \right\}$$

$$= a\left\{ \frac{x}{(1+a^2x)(1+\frac{x^2}{a^2})} + \frac{x^7}{(1+a^2x)(1+\frac{x^2}{a^2})(1+a^4x^4)(1+\frac{x^4}{a^4})} + \cdots \right\}$$

$$+ \frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1+a^2x)(1+\frac{x^2}{a^2})(1+a^2x^3)\cdots}$$

$$\times \frac{1+2x+2x^4+2x^7+\cdots}{(a-\frac{1}{a}) - (a^3-\frac{1}{a^3})x^2 + (a^5-\frac{1}{a^5})x^6 - \cdots}$$

$$\frac{1}{2}(1 - a^2x + a^2x^2 - a^4x^3 + \cdots)$$

$$= \frac{1}{1+a^2x} - \frac{a^4x^2}{(1+a^2x)(1+a^4x^2)}$$

$$\phi(x) = 1 + \frac{ax}{1-bx} + \frac{a^2x^3}{(1-bx)(1-bx^2)} + \cdots$$

$$= 1 + \frac{ax(1+ax^2)}{1-bx} + \frac{a^2x^5(b+ax)(1+ax^4)}{(1-bx)(1-bx^2)}$$

$$+ \frac{a^3x^{12}(b+ax)(b+ax^2)(1+ax^6)}{(1-bx)(1-bx^2)(1-bx^3)} + \cdots$$

$$= (1+ax) + \frac{ax^2(b+ax)(1+ax^3)}{1-bx}$$

$$+ \frac{a^2x^7(b+ax)(b+ax^2)(1+ax^4)}{(1-bx)(1-bx^2)} + \cdots$$

$$\phi(a) = (b+ax)\phi(ax) + 1-b$$

$$(1-ax)(1-ax^2)(1-ax^3)\cdots \left\{1 + \frac{ax}{1-x} + \frac{a^2x^4}{(1-x)(1-x^2)} + \cdots\right\}$$

$$= 1 - a^2x^2\,\frac{1-ax^2}{1-x} + a^4x^9\,\frac{(1-ax)(1-ax^4)}{(1-x)(1-x^3)} -$$

$$= (1-a^2x^2) - a^2x^3\,\frac{(1-ax)(1-a^2x^6)}{1-x} + a^4x^{11}\,\frac{(1-ax)(1-ax^4)}{(1-x)(1-x^2)(1-a^2x^{10})}$$

Cor. $(1-ax)(1-ax^2)(1-ax^3)\cdots \left\{1 + \frac{ax^2}{1-x} + \frac{a^2x^6}{(1-x)(1-x^2)} + \cdots\right\}$

$$= (1-ax) - a^2x^4\,\frac{(1-ax)(1-ax^3)}{1-x} + a^4x^{13}\,\frac{(1-ax)(1-ax^2)(1-ax^5)}{(1-x)(1-x^2)}$$

If $G(a,\lambda) = 1 + \frac{x(a+\lambda)}{(1-x)(1+\lambda x)} + \frac{x^3(a+\lambda)(a+\lambda x)}{(1-x)(1-x^2)(1+\lambda x)(1+\lambda x^2)} + \cdots$

then $\frac{G(ax,\lambda x)}{G(a,\lambda)} = \frac{1}{1 + \frac{ax+\lambda x^2}{1+}\frac{b x+\lambda x^2}{1+}\frac{a x^2+\lambda x^3}{1+}\frac{b x^2+\lambda x^4}{1+}\frac{a x^3+\lambda x^5}{1+}\cdots}$

$$\frac{1+x^2+x^{12}+\cdots}{1+x+x^3+\cdots} = 1 - x\cdot\frac{1-x}{1-x^4} + x^4\cdot\frac{(1-x)(1-x^3)}{(1-x^4)(1-x^5)} - \cdots$$

$$\frac{1+x+x^5+x^8+\cdots}{1+x+x^2+x^6+\cdots} = 1 - x^3\cdot\frac{1-x}{1-x^4} + x^8\cdot\frac{(1-x)(1-x^3)}{(1-x^4)(1-x^5)} - \cdots$$

$$\frac{1}{1+x+x^3+x^6+\cdots} = 1 - x\cdot\frac{1-x}{(1-x^2)^2} + x^4\cdot\frac{(1-x)(1+x^3)}{(1-x^2)^2(1-x^5)} - \cdots$$

Euler * $\frac{1+x-x^5-x^9+\cdots}{1+x+x^3+x^6+\cdots} = 1 - x^3\cdot\frac{1-x}{(1-x^2)^2} + x^8\cdot\frac{(1-x)(1-x^5)}{(1-x^2)^2(1-x^4)} - \cdots$

$$* \quad (1-x+x^2-x^3+\cdots)(1+x)(1+x^3)(1+x^5)\cdots$$

$$= 1 + \frac{2x^3}{1-x^4} + \frac{2x^8}{(1-x^2)(1-x^6)} + \frac{2x^{15}}{(1-x^2)(1-x^4)(1-x^{12})} + \cdots$$

$$\frac{1+x^4+x^{12}+\cdots}{1+x+x^3+\cdots} = 1 - x\cdot\frac{1-x}{1-x^2} + x^4\cdot\frac{(1-x)(1-x^3)}{(1-x^2)(1-x^4)} - \cdots$$

$$\frac{1+x^5+x^{12}+\cdots}{1+x^3+x^{11}+\cdots} = 1 - x^3\cdot\frac{1-x}{1-x^2} + x^5\cdot\frac{(1-x)(1-x^3)}{(1-x^2)(1-x^4)} - \cdots$$

$$1 + \frac{aq}{1+aq^2} + \frac{a^2q^6}{(1+aq^2)(1+aq^4)} + \frac{a^3q^{15}}{(1+aq^2)(1+aq^4)(1+aq^6)} + \cdots$$

$$= (1+aq) + a^3q^6\,\frac{1-q}{1+aq^2}(1+aq^5) + a^6q^{12}\,\frac{(1-q)(1-q^3)}{(1+aq^2)(1+aq^4)}(1+aq^9) + \cdots$$

$$1 + q + q^4 + q^6 + q^{10} + q^{12} + \cdots$$

$$G(a,b) = 1 + \frac{ax}{(1-x)(1+bx)} + \frac{a^2x^3}{(1-x)(1-x^2)(1+bx)(1+bx^2)} + \cdots$$

$$\frac{G(a,b)}{G(bx,b)} = 1 + ax\cdot\frac{G(bx,ax)}{G(bx,bx)} = 1 + \frac{ax}{1+bx}\cdot\frac{G(ax^2,bx)}{G(bx^2,bx^2)}$$

$$= 1 + \frac{ax}{1+bx} + \frac{ax^2}{1+bx^2} + \frac{ax^3}{1+bx^3} + \cdots$$

If $c_0 + c_1x + c_2x^2 + \cdots = \dfrac{(1-\alpha x)(1-\alpha' x)(1-\alpha'' x)\cdots}{(1-\beta x)(1-\beta' x)(1-\beta'' x)\cdots}$

then

$$c_0 + c_1x + c_2x^3 + c_3x^6 + \cdots$$

$$= (1+bx)(1+bx^2)(1+bx^3)\cdots \quad G(a,b)$$

$$G(a,b)/G(b,a) = \frac{(1+ax)(1+ax^2)(1+ax^3)\cdots}{(1+bx)(1+bx^2)(1+bx^3)\cdots}$$

(i) $$1 + \frac{ax}{1-x} + \frac{a^2x^4}{(1-x)(1-x^2)} + \frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)} + \cdots$$

$$= (1+ax)(1+ax^2)(1+ax^3)\cdots\left\{1 + \frac{ax}{(1-x^2)(1+ax)} + \frac{a^2x^6}{(1-x^2)(1-x^4)(1+ax)(1+ax^2)} + \cdots\right\}$$

(ii) $$= (1+ax^2)(1+ax^4)(1+ax^6)\cdots\left\{1 + \frac{ax}{(1-x^2)(1+ax^2)} + \frac{a^2x^4}{(1-x^2)(1-x^4)(1+ax)(1+ax^4)} + \cdots\right\}$$

$$G(a,b) = (1+ax)(1+ax^2)(1+ax^3)\cdots \times$$

$$1 - \frac{ab\,x^2}{(1-x)(1+ax)(1+bx)} + \frac{a^2b^2x^7}{(1-x)(1-x^2)(1+ax)(1+ax^2)(1+bx)(1+bx^2)} - \cdots$$

$$2 = \frac{1\cdot4}{2},\quad 7 = \frac{2\cdot7}{2},\quad 15 = \frac{3\cdot10}{2},\quad 26 = \frac{4\cdot13}{2},\ \ldots$$

If $|a| \geq |b|$ and $|x| \leq 1$, then

$$\frac{1}{a+c} - \frac{ab}{a+b+cx} - \frac{ab}{a+b+cx^2} - \frac{ab}{a+b+cx^3} - \cdots$$

$$= \frac{1}{c-b+a} + \frac{bc}{c-b+\frac{a}{x}} + \frac{bc}{c-b+\frac{a}{x^2}} + \cdots$$

$$1 + \frac{a}{1} + \frac{b}{x} + \frac{a}{1} + \frac{b}{x^2} + \cdots + \frac{b}{x^{n-1}} + \frac{a}{1} + \cdots$$

$$= 1 + a - \frac{ab}{a+b+x} - \frac{ab}{a+b+x^2} - \cdots - \frac{ab}{a+b+x^n} -$$

$$1 + \frac{a^2x^4}{1-x^4} + \frac{a^4x^{16}}{(1-x^4)(1-x^8)} + \cdots$$

(iii) $$= (1-ax)(1-ax^2)(1-ax^3)\cdots\left\{1 + \frac{ax}{(1-x^2)(1-ax)} + \frac{a^2x^4}{(1-x^2)(1-x^4)(1-ax)(1-ax^2)} + \cdots\right\}$$

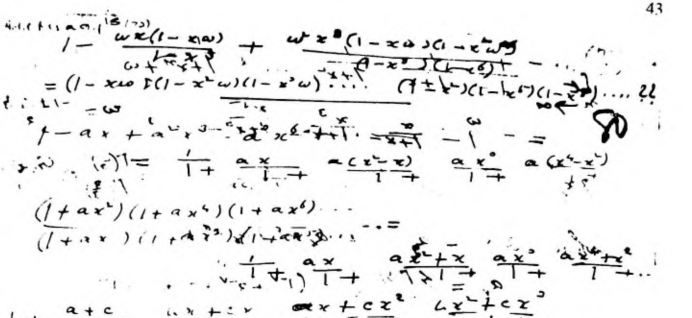
$$\frac{(1-ax)(1-ax^2)(1-ax^3)\cdots}{(1+bx)(1+bx^2)(1+bx^3)\cdots}\left\{1 + \frac{x}{1-x}\cdot\frac{b+ax}{1+cx} + \frac{x^2}{(1-x)(1-x^2)}\cdot\frac{(b+ax)(b+ax^2)}{(1+cx)(1+cx^2)} + \cdots\right\}$$

$$= (1-ax) - x^2\cdot\frac{1-ax}{1-x}\cdot\frac{(b+ax)(c+ax)}{(1+bx)(1+cx)}(1-ax^3) + \cdots$$

$$+ x^7\cdot\frac{(1-ax)(1-ax^2)}{(1-x)(1-x^2)}\cdot\frac{(b+ax)(b+bx^2)(c+ax)(c+ax^2)}{(1+bx)(1+bx^2)(1+cx)(1+cx^2)}(1-ax^5) - \cdots$$

$$= (1-ax)(1-ax^2)(1-ax^3)\cdots$$

$$\times\left\{1 + \frac{x^2(a-bc)}{(1-x)(1+bx)(1+cx)} + \frac{x^6\cdot(a-bc)(a-bcx)}{(1-x)(1-x^2)(1+bx)(1+bx^2)(1+cx)(1+cx^2)} + \cdots\right.$$



$$\frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2+ax}{1+}\ \frac{x^4+bx^2}{1+}\ \frac{x^6+ax^3}{1+}\ \cdots$$

$a = 0, \pm 1$; $b = 0, \pm 1$. 2 cases

Order 4 (i) $a = 0$, $b = 1$

$$\frac{(1-x)(1-x^5)(1-x^9)\cdots}{(1-x^4)(1-x^6)(1-x^{10})\cdots} \cdot \frac{(1-x^3)(1-x^7)(1-x^{11})}{(1-x^2)(1-x^6)(1-x^{10})}$$

$$= \frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2+x}{1+}\ \frac{x^3}{1+}\ \frac{x^6+x^2}{1+}\ \cdots$$

(iii) $a = 0$, $b = -1$

$$1 - x + x^3 - x^6 + \cdots = \frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2-x}{1+}\ \frac{x^3}{1+}\ \frac{x^4-x^2}{1+}\ \cdots$$

Order 5 (iii) $a = 0$, $b = 0$.

$$\frac{(1-x)(1-x^4)\cdots}{(1-x^2)(1-x^7)\cdots} \cdot \frac{(1-x^4)(1-x^8)\cdots}{(1-x^3)(1-x^8)\cdots} = \frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2}{1+}\ \cdots$$

Order 6 (iv) $a = 1$, $b = 1$

$$\frac{(1-x)(1-x^7)(1-x^{13})\cdots}{(1-x^3)(1-x^8)(1-x^{15})\cdots} \cdot \frac{(1-x^5)(1-x^{11})(1-x^{17})\cdots}{(1-x^5)(1-x^9)(1-x^{13})\cdots}$$

$$= \frac{1}{1+}\ \frac{x+x^2}{1+}\ \frac{x^2+x^4}{1+}\ \frac{x^3+x^6}{1+}\ \cdots$$

(v) $a = -1$, $b = 1$ change x to $-x$ in (iv)

(vi) $a = -1$, $b = -1$

$$1 + x - x^5 - x^8 + x^{16} + x^{21} - x^{33} - \cdots = \frac{1}{1+}\ \frac{x^2-x}{1+}\ \frac{x^4-x^2}{1+}\ \frac{x^6-x^3}{1+}\ \cdots$$

(vii) $a = 1$, $b = -1$ change x to $-x$ in (vi).

Order 8 (viii) $a = 1$, $b = 0$

$$\frac{(1-x)(1-x^9)(1-x^{17})\cdots}{(1-x^3)(1-x^{11})(1-x^{19})\cdots} \cdot \frac{(1-x^7)(1-x^{15})(1-x^{23})\cdots}{(1-x^5)(1-x^{13})(1-x^{21})\cdots}$$

$$= \frac{1}{1+}\ \frac{x^2+x}{1+}\ \frac{x^5}{1+}\ \frac{x^6+x^3}{1+}\ \cdots$$

(ix) $a = -1$, $b = 0$, change x to $-x$ in (viii)

$$1 - x + x^3 - x^6 + \cdots = \frac{1}{1+}\ \frac{x+x^2}{1+}\ \frac{x^3+x^4}{1+}\ \frac{x^5+x^6}{1+}\ \cdots$$

$$\frac{(1+x^3)(1+x^7)(1+x^{11})\cdots}{(1+x)(1+x^5)(1+x^9)\cdots} = \frac{1}{1+}\ \frac{x}{1+}\ \frac{x^3+x^2}{1+}\ \frac{x^5}{1+}\ \frac{x^7+x^4}{1+}\ \cdots$$

$$1 - x + x^2 - x^5 + x^7 - \cdots = \frac{a}{2+}\ \frac{x+x}{1+}\ \frac{x^2+x^3}{1+}\ \frac{x^3+x^5}{1+}\ \cdots$$

Let $\frac{1}{1-}\ \frac{1}{1+x-}\ \frac{1}{1+x^2-}\ \cdots\ \frac{1}{1+x^{2n}+\omega}$
$n\to\infty$

$$= -\frac{\omega}{1-}\ \frac{\omega}{1+x-}\ \frac{x^3}{1+x^2-}\ \frac{x^4}{1+x^3-}\ \cdots, \qquad \omega = \frac{-1\pm i\sqrt{3}}{2}$$

$$(3x)^{\frac{1}{3}}\ \frac{1}{1+e^{3x}}\ \frac{1}{1+e^{3x}-}\ \frac{1}{1+e^{3x}-}\ \cdots = \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\, e^{G(x)}$$

where $G(x) = a_2 x^2 + a_4 x^4 + a_6 x^6 + \ldots$,

$$a_\nu = \frac{4\,\Gamma(\nu)\,x^\nu}{(2\pi/\sqrt{3})^{3\nu+1}}\,(1^{-\nu} + 2^{-\nu} + 3^{-\nu} + \ldots)(1^{-\nu-1} - 2^{-\nu-1} + 4^{-\nu-1} - \ldots)$$

$a_2 = \frac{1}{10}$, $a_4 = \frac{1}{4320}$, $a_6 = \frac{1}{38880}$,

minimum term $\sim \frac{3}{\pi}\sqrt{(\frac{3x}{\pi})}\, e^{-\frac{4\pi^2}{3x}}$

$$u_\lambda = \frac{1}{1+e^{(\lambda+1)x}-}\ \frac{1}{1+e^{(\lambda+2)x}-}\ \cdots$$

$$u_\lambda + \frac{1}{u_{\lambda-1}} = 1 + e^{\lambda x}$$

$$u_\lambda = \frac{1}{\lambda} - \frac{\phi_0}{\lambda^2} + x\left\{\frac{\lambda+1}{2} + \frac{\phi_1 + (\lambda^2-1)\left(\frac{1}{12}-\frac{\phi_0}{2}\right)}{1-\lambda\phi_0}\right.$$

$$\left. + x^2\left[\frac{\lambda(\lambda+1)(\lambda+2)}{6} - \frac{\phi_2 + \lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{720}-\frac{\lambda\phi_0}{24}\right) + \left(\lambda+\frac{1}{2}\lambda^3\phi_0\right)\left(\frac{1}{12}-\frac{\phi_1}{2}\right)}{(1-\lambda\phi_0)^2}\right.\right.$$

$$\left.\left. + \frac{\lambda}{1-\lambda\phi_0}\left\{\frac{1}{12}-\frac{\phi_1}{2} + \lambda^2\left(\frac{1}{720}-\frac{\lambda\phi_0}{24}\right)\right\}^2\right] + \cdots\right.$$

where $\phi_0, \phi_1, \phi_2, \ldots$ are arbitrary and independent of x. If $x = 0$, then $\phi_1, \phi_2, \phi_3, \ldots$ all vanish and $\phi_0 = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}$.

Let $\frac{1}{1+x-}\ \frac{1}{1+x^2-}\ \frac{1}{1+x^3-}\ \cdots\ \frac{1}{1+x^{2n}+a}$
$n\to\infty$

$$= -\omega^2\,\frac{\Omega - \omega^{n+1}}{\Omega - 1}\cdot\frac{(1-x^2)(1-x^4)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots}$$

where $\Omega = \frac{1-a\omega^2}{1-a\omega}\cdot\frac{(1-x\omega^2)(1-x^2\omega^2)(1-x^3\omega^2)\cdots}{(1-x\omega)(1-x^2\omega)(1-x^3\omega)\cdots}$

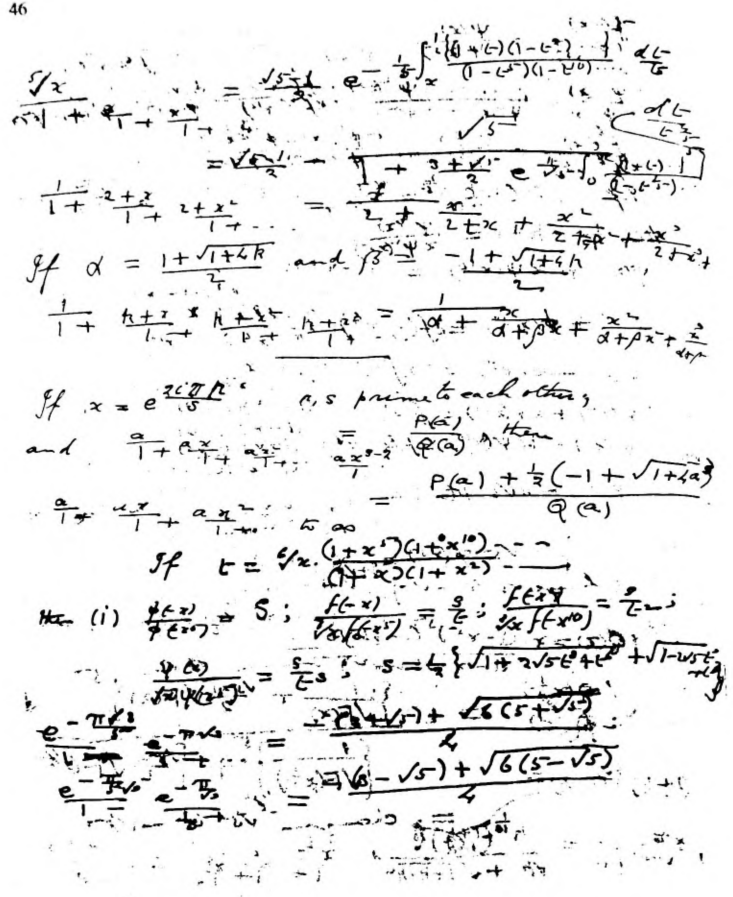
numerator and denominator can be equated separately

$$\frac{2\sqrt{x}}{1-}\ \frac{1}{e^{2x}+e^{-2x}-}\ \frac{1}{e^{2x}+e^{-2x}-}\ \frac{1}{e^{2x}+e^{-2x}-}\ \cdots = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\, e^{G(x)}$$

where $G(x) = a_2 x^2 + a_6 x^6 + a_{10} x^{10} + \ldots$,

$$a_\nu = \frac{4\,\Gamma(\nu)\,x^\nu}{\pi^{2\nu+1}}\,(1^{-\nu} + 3^{-\nu} + 5^{-\nu} + \ldots)(1^{-\nu-1} - 3^{-\nu-1} + 5^{-\nu-1} - \ldots)$$

$a_2 = \frac{1}{24}$, $a_6 = \frac{1}{1152}$, $\ldots$; minimum term $\sim \frac{2}{\pi}\sqrt{(\frac{2x}{\pi})}\, e^{-\frac{\pi^2}{2x}}$



(0) $\frac{f^3(x^2)}{f(-x)} = \left(\frac{1}{1-x} + \frac{x}{1-x^4} + \frac{x^2}{1-x^7} + \cdots\right) - \left(\frac{x}{1-x^2} + \frac{x^3}{1-x^6} + \frac{x^5}{1-x^{10}} + \cdots\right)$

$u = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x^2)(1-x^3)} + \cdots$; $v = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x^2)(1-x^3)} + \cdots$

(1) $= \left(\frac{1}{1-x^2} + \frac{x}{1-x^7} + \frac{x^2}{1-x^{12}} + \cdots\right) - \left(\frac{x^2}{1-x^3} + \frac{x^6}{1-x^8} + \frac{x^{10}}{1-x^{13}} + \cdots\right)$

$\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^2 u$ $\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^2 v$

(2) $= \left(\frac{1}{1-x} + \frac{x^3}{1-x^6} + \frac{x^6}{1-x^{11}} + \cdots\right) - \left(\frac{x}{1-x^4} + \frac{x^3}{1-x^9} + \frac{x^5}{1-x^{14}} + \cdots\right)$

$\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^2 \frac{u^2}{v}$

(3) $= \left(\frac{1}{1-x^2} + \frac{x}{1-x^9} + \frac{x^2}{1-x^{11}} + \cdots\right) - \left(\frac{x^3}{1-x^4} + \frac{x^7}{1-x^9} + \frac{x^{11}}{1-x^{14}} + \cdots\right)$

$\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^2 \frac{v^2}{u}$

(4) $= \left(\frac{1}{1-x^2} + \frac{x^2}{1-x^7} + \frac{x^6}{1-x^{12}} + \cdots\right) - \left(\frac{x^2}{1-x^3} + \frac{x^6}{1-x^8} + \frac{x^7}{1-x^{13}} + \cdots\right)$

(0) $= \frac{1+x}{1-x} + x^5 \cdot \frac{1+x^4}{1-x^4} + x^{16} \cdot \frac{1+x^7}{1-x^7} + \cdots$

$-\left(x \cdot \frac{1+x^2}{1-x^2} + x^8 \cdot \frac{1+x^5}{1-x^5} + x^{21} \cdot \frac{1+x^8}{1-x^8} + \cdots\right)$

(1) $= \left(\frac{1}{1-x} + \frac{x^2}{1-x^6} + \frac{x^4}{1-x^{11}} + \cdots\right) - \left(\frac{x^2}{1-x^4} + \frac{x^5}{1-x^9} + \frac{x^8}{1-x^{14}} + \cdots\right)$

(2) $= \left(\frac{1}{1-x^3} + \frac{x}{1-x^8} + \frac{x^2}{1-x^{13}} + \cdots\right) - \left(\frac{x}{1-x^2} + \frac{x^3}{1-x^7} + \frac{x^5}{1-x^{12}} + \cdots\right)$

(3) $= \frac{1+x}{1-x} + x^7 \cdot \frac{1+x^6}{1-x^6} + x^{24} \cdot \frac{1+x^{11}}{1-x^{11}} \cdots$

$-\left(x^3 \cdot \frac{1+x^5}{1-x^4} + x^{16} \cdot \frac{1+x^9}{1-x^9} + x^{39} \cdot \frac{1+x^{14}}{1-x^{14}} \cdots\right)$

(4) $= \frac{1+x^2}{1-x^2} + x^9 \cdot \frac{1+x^7}{1-x^7} + x^{28} \cdot \frac{1+x^{12}}{1-x^{12}} + \cdots$

$-\left(x \cdot \frac{1+x^3}{1-x^3} + x^{12} \cdot \frac{1+x^8}{1-x^8} + x^{33} \cdot \frac{1+x^{13}}{1-x^{13}} - \right)$

$1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \cdots = \frac{(2)}{(1)} = \sqrt{\frac{(2)}{(1)}} = \sqrt[3]{\frac{(2)}{(3)}}$.

(1) $= \left(\frac{1}{1-x} + \frac{x^4}{1-x^{11}} + \frac{x^8}{1-x^{21}} + \cdots\right) - \left(\frac{x^6}{1-x^9} + \frac{x^{11}}{1-x^{19}} + \frac{x^{19}}{1-x^{29}} \cdots\right)$

(2) $= \left(\frac{1}{1-x^3} + \frac{x^2}{1-x^{13}} + \frac{x^4}{1-x^{23}} + \cdots\right) - \left(\frac{x^5}{1-x^7} + \frac{x^{13}}{1-x^{17}} + \frac{x^{21}}{1-x^{27}} \cdots\right)$

~~(3)~~ If $\frac{K'}{K} = \sqrt{47}$ and $2^{\frac{1}{8}}(kk')^{\frac{1}{12}} = x$

(i) $1 - \frac{x}{1} - \frac{x^2}{1-x^3} - \frac{x^5}{1-x^4} = 0$. $\sqrt{23}$. $1 - \frac{x^2}{1-x^3} - \frac{x^5}{1-x^4} = 0$

If $\frac{K'}{K} = \sqrt{89}$, $\frac{L'}{L} = \sqrt{\frac{13}{3}}$ $\left(\frac{kk'}{\varepsilon\varepsilon'}\right)^{\frac{1}{12}} = x$

(ii) $1 - \frac{x}{1} - \frac{x^2}{1-x^3} - \frac{x^5}{1-x^4} = 0$ $\sqrt{31}$. $1 - \frac{x}{1-x^3} - \frac{x^5}{1-x^4} = 0$

(i) $x = \sqrt{2} \cdot e^{-\frac{\pi}{12}\sqrt{47}} (1 - e^{-\pi\sqrt{47}})(1 + e^{-2\pi\sqrt{47}})(1 - e^{-3\pi}$

(ii) $x = e^{-\frac{\pi}{12}\sqrt{\frac{13}{3}}} (1 + e^{-\pi\sqrt{\frac{13}{3}}})(1 + e^{-5\pi\sqrt{\frac{13}{3}}}) \cdots$ $\times (1 + e^{-7\pi\sqrt{\frac{13}{3}}})$

If $1 - \frac{ax}{1-} - \frac{ax^2}{1-} - \frac{ax^3}{1-} \cdots = 0$ then $x = \frac{1}{a+1 - \frac{1}{a+3 - \frac{1}{a+5 - \frac{1}{a+7 - \cdots}}}}$ (59)

If $1 - \frac{x}{1-} - \frac{x^2}{1-} - \frac{x^3}{1-} \cdots = 0$, then $x = .5762\ldots$

$$x = \frac{2}{a-1+\sqrt{(a+1)(a+5)}} + O\left(\frac{1}{a^8}\right)$$

$$x = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5279}{a^9} - \frac{22573}{a^{10}} + \cdots$$

$$\frac{2}{a-1+\sqrt{(a+1)(a+5)}} = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1265}{a^8} + \frac{5275}{a^9} - \frac{22431}{a^{10}} + \cdots$$

$$x = \frac{1}{\frac{a-1+\sqrt{(a+1)(a+5)}}{2}} + \left(\frac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3$$

$$\frac{1}{\sqrt{3}} = .57735 \qquad \frac{1}{5\left(\frac{2}{9}\sqrt{3}-1\right)} =$$

$$(1-x^2)(1-x^4)(1-x^6)\cdots \qquad w = f(-x^{11}, -x^{17}) + x f(-x^7, -x^{23})$$

$$(1-x^3)(1-x^6)(1-x^9)\cdots \qquad w = f(-x^{11}, -x^{17}) + x^2 f(-x^7, -x^{23})$$

$$3 + 7x - 13x^4 - 17x^7 + \cdots = \left(\frac{3}{e^2} + e^3\right)\left[(1-x^5)(1-x^{10})\cdots\right]^{\frac{3}{5}} x^{\frac{1}{5}} -$$

$$1 + 9x^2 - 11x^3 - 17x^9 + \cdots = \left(\frac{1}{e^3} - 3e^2\right)\left[(1-x^5)(1-x^{10})\cdots\right]^{\frac{9}{5}} x^{\frac{9}{5}} -$$

$$(3 + 7x - \cdots)(1 + 9x^2 - \cdots) = 3 f(-x) + x^2 f'(-x)$$

$$\frac{w^3}{w^2}\left[(1-x^5)(1-x^{10})\cdots\right]^2 = 1 + 3\left(\frac{x}{1-x} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \cdots\right) + \left(\frac{x^2}{1-x^2} - \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} - \cdots\right)$$

$$x\frac{w^3}{w^2}\left[(1-x^5)(1-x^{10})\cdots\right]^2 = \frac{x}{1-x} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \cdots - 3\left(\frac{x^2}{1-x^2} - \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} - \cdots\right)$$

$$e = \frac{1/x}{1 + \frac{11}{1-} \frac{x^2}{1-}}$$

$$\frac{1 - 9x^2 + 11x^3 - 19x^5 + \cdots}{1 + x^2 + x^3 + x^5 + \cdots} = 1 - 10\left(\frac{x^2}{1+x^2} - \frac{x^3}{1+x^3} + \frac{x^7}{1+x^7} - \cdots\right)$$

$$\frac{3 - 7x + 13x^5 - 17x^7 + \cdots}{1 + x^2 + x^4 + x^7} = 3 - 10\left(\frac{x}{1+x} - \frac{x^4}{1+x^4} + \frac{x^6}{1+x^6} - \cdots\right)$$

$$\frac{1 - 3\frac{x}{a} + 5a x^3 - 7\frac{x^6}{a^2} + 9a^2 x^{10} - \cdots}{1 + \frac{x^2}{a} + a x^3 + \frac{x^6}{a^2} + a^2 x^{10} + \cdots}$$

$$= 1 - 4\left(\frac{x}{a+x} - \frac{x^3}{\frac{1}{a}+x^3} + \frac{x^5}{a+x^5} - \frac{x^7}{\frac{1}{a}+x^7} + \cdots\right)$$

$$\left\{(1-x^{10})(1-x^{20})\cdots\right\}^3 = f^3(-x^{11}, -x^{17}) + x^3 f^3(-x^7, -x^{23})$$

$$+ (1-x^6)(1-x^{12})(1-x^{18})\cdots = V f(-x^6)\frac{f^3(-x^{10})}{f(-x^{30})}$$

$$f^3(-x^{11}, -x^{19}) + x^2 f^3(-x^7, -x^{23}) = V f(-x^6)\frac{f^3(-x^{10})}{f(-x^{30})}$$

$$V = 1 + \frac{x^3}{1-x^3} + \frac{x^{12}}{(1-x^3)(1-x^4)} + \cdots \qquad V = 1 + \frac{x^6}{1-x^6} + \frac{x^{10}}{(1-x^5)(1-x^6)} + \cdots$$

$$\frac{1}{1+\frac{x}{1+}\frac{x^2}{1+}\cdots} = u_0 + u_1 x + u_2 x^2 + \cdots$$

$$1 + \frac{x}{1+}\frac{x^2}{1+}\cdots = v_0 + v_1 x + v_2 x^2 + \cdots$$

n	u_n	v_n	n	u_n	v_n	n	u_n	v_n	n	u_n	v_n	n	u_n	v_n
1	−1	1	21	−7	5	41	−33	24	61	−119	88	81	−366	270
2	1	0	22	5	−3	42	23	−14	62	83	−53	82	254	−165
3	−1	−1	23	0	−8	43	−2	−37	63	−8	−121	83	−25	−407
4	−1	0	24	−5	−3	44	−23	−8	64	−79	−27	84	−288	−82
5	1	1	25	9	8	45	−41	37	65	143	191	85	439	397
6	−1	1	26	−10	7	46	−45	33	66	−157	115	86	−473	348
7	1	−1	27	7	−5	47	32	−21	67	110	−73	87	328	−216
8	0	−2	28	−1	−19	48	−4	−52	68	−12	−178	88	−35	−528
9	−1	0	29	−7	−2	49	−30	−10	69	−103	−35	89	−305	−104
10	2	2	30	14	13	50	57	53	70	191	177	90	581	679
11	−3	2	31	−16	12	51	−64	47	71	−212	156	91	−616	453
12	2	−1	32	11	−7	52	45	−29	72	147	−95	92	436	−277
13	0	−3	33	−1	−18	53	−4	−72	73	−14	−236	93	−43	−682
14	−2	−1	34	−11	−4	54	−63	−15	74	−199	−68	94	−397	−123
15	4	3	35	20	18	55	78	71	75	253	232	95	720	664
16	−4	3	36	−22	16	56	−86	63	76	−277	204	96	−789	580
17	3	−2	37	16	−11	57	60	−40	77	193	−127	97	226	−265
18	−1	−5	38	−2	−26	58	−7	−98	78	−22	−311	98	−59	−875
19	−3	−1	39	−15	−5	59	−57	−19	79	−100	−61	99	−304	−172
20	6	6	40	29	27	60	107	99	80	372	309	100	923	855

$u_{175} = 28000$

$$u_{1000} = +599312191476\,5$$
$$u_{1001} = -635043849283\,21$$
$$u_{1002} = +425092952570$$
$$u_{1003} = -\ 42579265368\,04$$
$$u_{1004} = -369697734\,4774$$

$$v_{1000} = +561012833077$$
$$v_{1001} = +474699\,9\,2\,87\,52$$
$$v_{1002} =$$

$$\frac{1}{1+\frac{x}{1+}\frac{x^2}{1+}\cdots} = \frac{(1-x^4-x^{11}+x^{19}+x^{31}-\cdots)-x(1-x^5-x^{17}+x^{29}+x^{43}-\cdots)}{1-x^5-x^{10}+x^{25}+x^{31}-\cdots}$$

$$1+\frac{x}{1+}\frac{x^2}{1+}\cdots = \frac{(1-x^7-x^8+x^{29}+x^{31}-\cdots)+x(1-x^2-x^{10}+x^{19}+x^{47}-\cdots)}{1-x^5-x^{10}+x^{25}+x^{31}-\cdots}$$

$$\frac{1}{1+\frac{x}{1+}\frac{x^2}{1+}\cdots} = u_0+u_1x+u_2x^2+\cdots$$

Addition 94 (67)

$$1+\frac{x}{1+}\frac{x^2}{1+}\cdots = v_0+v_1x+v_2x^2+\cdots$$

$$u_0+u_5x+u_{10}x^2+\cdots = \frac{f(-x^{35},-x^{45})+x^3f(-x^{15},-x^{55})}{f(-x)}$$

$$u_1+u_6x+u_{11}x^2+\cdots = -\frac{f(-x^{33},-x^{66})+x^2f(-x^{99},-x^{57})}{f(-x)}$$

$$u_2+u_7x+u_{12}x^2+\cdots = -\frac{x^3f(-x^{14},-x^{77})-xf(-x^{11},-x^{66})}{f(-x)}$$

$$u_3+u_8x+u_{13}x^2+\cdots = \frac{f(-x^{99},-x^{77})+x^7f(-x^{33},-x^{55})}{f(-x)}$$

$$u_4+u_9x+u_{14}x^2+\cdots = -\frac{f(-x^{26},-x^{74})+x^8f(-x^{78},-x^{52})}{f(-x)}$$

$$v_0+v_5x+v_{10}x^2+\cdots = \frac{f(-x^{39},-x^{65})+x^4f(-x^{13},-x^{91})}{f(-x)}$$

$$v_1+v_6x+v_{11}x^2+\cdots = \frac{f(-x^{55},-x^{49})+x^6f(-x^{77},-x^{27})}{f(-x)}$$

$$v_2+v_7x+v_{12}x^2+\cdots = -x\cdot\frac{f(-x^{22},-x^{154})-x^7f(-x^{66},-x^{110})}{f(-x)}$$

$$v_3+v_8x+v_{13}x^2+\cdots = -\frac{f(-x^{143},-x^{77})+xf(-x^{55},-x^{165})}{f(-x)}$$

$$v_4+v_9x+v_{14}x^2+\cdots = -x^2\frac{f(-x^{187},-x^{143})-x^5f(-x^{55},-x^{209})}{f(-x)}$$

Product

$$u_0+u_2x+u_4x^2+\cdots = \frac{f(-x^{10})}{f(-x^5)}\cdot\frac{f(-x^{40})}{f(-x^{20})}\cdot\frac{f(-x^{15})}{f(-x^{30})}\cdot\frac{f(-x^{60})}{f(-x^{60})}\cdots$$

$$u_1+u_3x+u_5x^2+\cdots = \frac{f(-x^2)}{f(-x^{10})}\cdot\frac{f(-x^{20})}{f(-x^{40})}\cdot\frac{f(-x^{30})}{f(-x^{30})}\cdot\frac{f(-x^{60})}{f(-x^{60})}\cdots$$

$$v_0+v_1x+v_2x^2+\cdots = \frac{f(-x^5)}{f(-x)}\cdot\frac{f(-x^{20})}{f(-x^{10})}\cdot\frac{f(-x^{30})}{f(-x^{30})}\cdot\frac{f(-x^{60})}{f(-x^{60})}\cdots$$

$$v_1+v_3x+v_4x^2+\cdots = \frac{1}{f(-x)}\cdot\frac{f(-x^{10})}{f(-x^5)}\cdot\frac{f(-x^{30})}{f(-x^{30})}\cdot\frac{f(-x^{60})}{f(-x^{60})}\cdots$$

$$f^2(-x^2) = \{\psi^2(x)-x\psi^2(5)\}\{\psi^2(5)\psi^2(x)-5\psi^2(25)\} = \{\psi^2(x)-\phi^2(5)\}\{\phi^2(5)-5\phi^2(25)\}$$

$$f^2(-x^2)f^2(-x^{10}) = \{\phi^2(x)-\phi^2(5)\}\{\phi^2(5)-5\phi^2(25)\}$$

$$\frac{f^{10}(-x)}{f^2(-x^2)} - 125x^2\frac{f^{10}(-x^{25})}{f^2(-x^{50})} = \{1+240(\cdots)\}\{\cdots+1\}$$

$$1+240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\cdots\right) = \frac{f^{10}(-x)}{f^2(-x^2)}+250x\frac{f^{10}(-x^{25})}{f^2(-x^{50})}+3125x^2\frac{f^{10}(-x^{125})}{f^2(-x^{250})}$$

$$1+240\left(\frac{x}{1-x}+\frac{2^3x^2}{1-x^2}+\cdots\right) = \frac{f^{10}(-x)}{f^2(-x^2)}+10x\frac{f^{10}(-x^5)}{f^2(-x^{10})}+5x^2\frac{f^{10}(-x^{25})}{f^2(-x^{50})}+\cdots$$

$$\nu = x \cdot \frac{f^3(-x) f^2(\epsilon x^{1/3})}{f^3(\epsilon^2 x) f^2(\epsilon x^{1/3})}$$

$$\int_0^x f(t) f(\epsilon t^3) f(\epsilon^2 t^3) f(\epsilon t^{1/3}) f(\epsilon^2 t^{1/3})\, dt = \frac{1}{3} \int_{2\tan^{-1}\frac{1}{\sqrt{3}}}^{2\tan^{-1}\frac{1}{\sqrt{3}}\sqrt{\frac{1-11\nu-\nu^2}{1+\nu-\nu^2}}} \frac{d\phi}{\sqrt{1 - \frac{3}{4}\sin^2\phi}}$$

$$= \frac{1}{9} \int_{2\tan^{-1}\frac{1-\nu\left(\frac{\sqrt{5}-1}{2}\right)^3}{1+\nu\left(\frac{\sqrt{5}-1}{2}\right)^3}\sqrt{\frac{1+\nu\frac{\sqrt{5}+1}{2}}{1-\nu\cdot\frac{\sqrt{5}+1}{2}}\cdot\frac{1-\nu\left(\frac{\sqrt{5}+1}{2}\right)^5}{1+\nu\left(\frac{\sqrt{5}+1}{2}\right)^5}}}^{\tan^{-1}(3-\sqrt{5})} \frac{d\phi}{\sqrt{1 - \frac{15}{16}\sin^2\phi}}$$

$$= \frac{1}{4} \int_{\tan^{-1}(3-\sqrt{5})}^{\tan^{-1}(3-\sqrt{5})\sqrt{\frac{1-\nu\frac{\sqrt{5}-1}{2}}{1+\nu\frac{\sqrt{5}+1}{2}}\cdot\frac{1-\nu\left(\frac{\sqrt{5}+1}{2}\right)^5}{1+\nu\left(\frac{\sqrt{5}+1}{2}\right)^5}}} \frac{d\phi}{\sqrt{1 - \frac{15}{16}\sin^2\phi}}$$

$$\nu = x \frac{f_1(\epsilon x^4) f_1(\epsilon^2 x^4)}{f_1(\epsilon x^2) f_1(\epsilon^2 x^2)}$$

$$\int_0^x f(t) f(\epsilon t^4) f(\epsilon^2 t^4) f(\epsilon t^{1/4}) f(\epsilon^2 t^{1/4})\, dt = \cdots \int \frac{d\phi}{\sqrt{1 - \frac{16\sqrt{13}-13}{27\sqrt{13}}\sin^2\phi}}$$

$$\cos^{-1}\frac{1}{7}\cdot\frac{1+\nu^2}{1-\nu}\sqrt{13+16\sqrt{13}}$$

$$Q = 1 + 240\left(\frac{x^4}{1-x^4} + \frac{2^4 x^8}{1-x^8} + \cdots\right)$$

$$R = 1 - 504\left(\frac{x^4}{1-x^4} + \frac{2^4 x^8}{1-x^8} + \cdots\right)$$

$$\int_{e^{-2\pi}}^{x} \sqrt{Q}\, dx = \frac{Q\sqrt{Q} - R}{6\sqrt{Q} + R}$$

$$\left\{\begin{array}{l} \frac{R^2}{Q^3} = z, \text{ where} \\ \frac{dz}{z} = -\frac{Q^3 - R^2}{QR} \end{array}\right.$$

$$1 - 504\left(\frac{x^4}{1-x^4} + \cdots\right) = \left\{\frac{f^{16}(-x)}{f^{16}(x^2)} - 500x\, f^8(x) f^8(-x) - 1536 x^2 f^8(x) f^8(x^2)\right\}$$

$$\sqrt{1 + 224x\frac{f^8(x^2)}{f^8(x)} + 1296x^2 \frac{f^{16}(x^2)}{f^{16}(x)}}$$

$$1 - 504\left(\frac{x^4}{1-x^4} + \cdots\right) = \left\{\frac{f^{16}(-x)}{f^{16}(x^2)} + 4x f^8(x) f^8(-x)\right\}$$

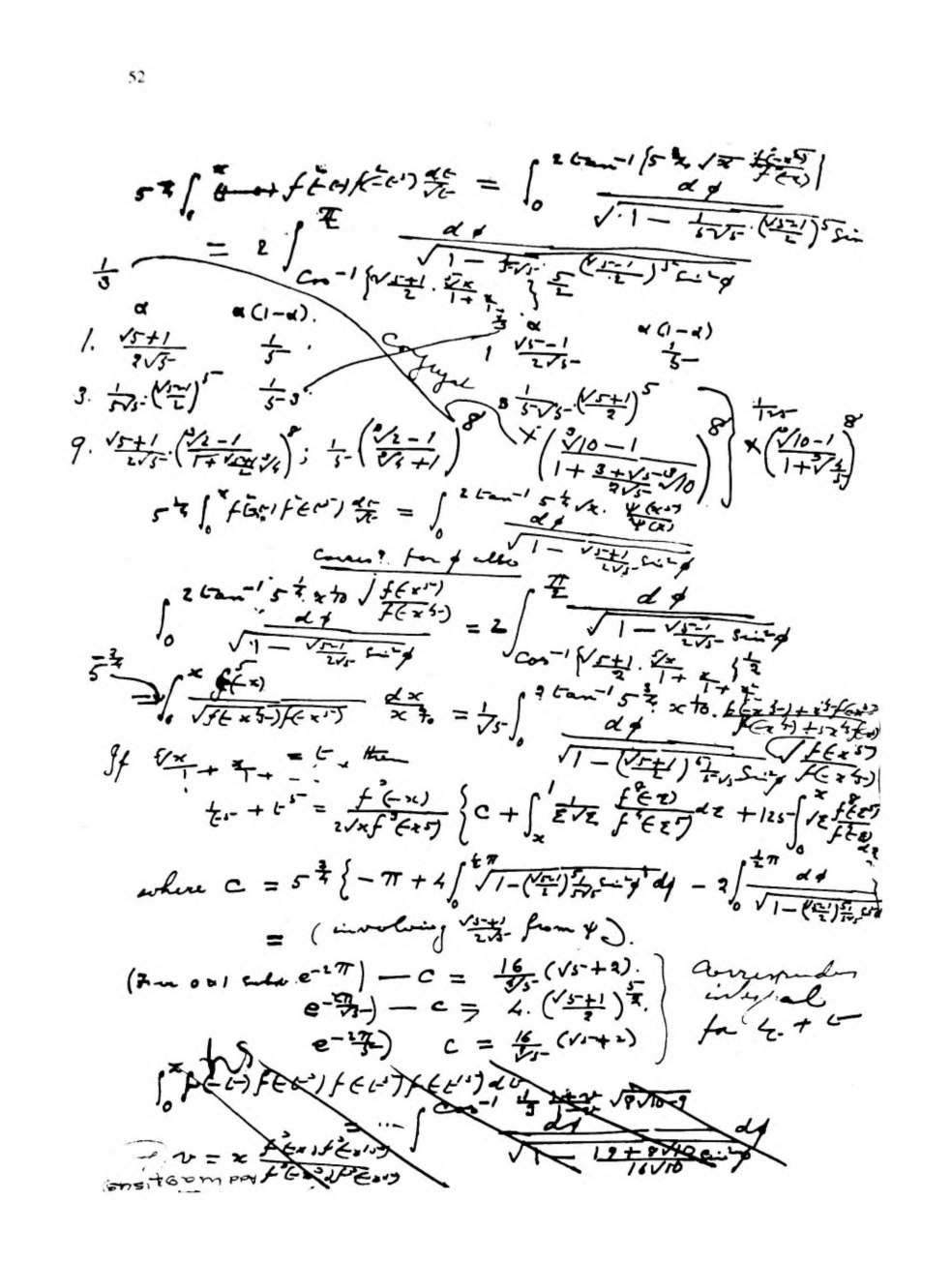
$$- x^2 f^8(x) f^8(x^2)\Big\}\sqrt{1 + 224x\frac{f^8(x^2)}{f^8(x)} + 1296x^2\frac{f^{16}(x^2)}{f^{16}(x)}}$$

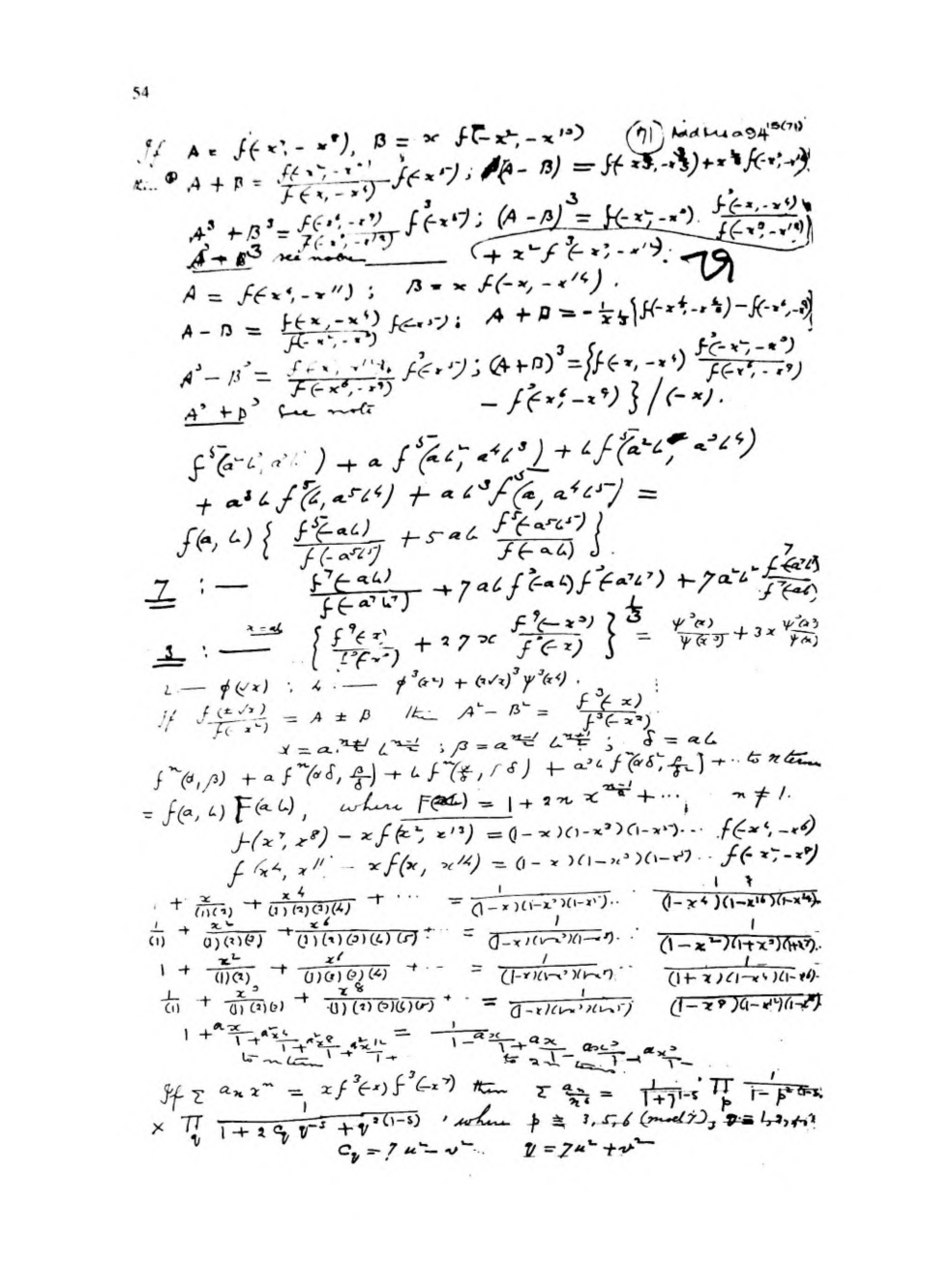
$$A = 1 + 240\left(\frac{x^4}{1-x^4} + \cdots\right) \qquad B = 1 + 240\left(\frac{x^8}{1-x^8} + \cdots\right)$$

$$\sqrt{A^2 + 976AB + 625B^2} = 12\sqrt{5}\left\{\frac{f^{16}(-x)}{f^{16}(x^2)} + 26x f^8(x) f^8(-x) + 125x^2 \frac{f^{16}(x^2)}{f^{16}(x)}\right\}$$

$$A = 1 - 504\left(\frac{x^4}{1-x^4} + \cdots\right) \qquad B = 1 - 504\left(\frac{x^8}{1-x^8} + \cdots\right)$$

$$\sqrt{5(A + 125B)^2 - 196^2 AB} = 25\sqrt{2}\left\{\frac{f^{16}(-x)}{f^{16}(x^2)} + 6x f^8(x) f^8(-x) + 125x^2 \frac{f^{16}(x^2)}{f^{16}(x)}\right\}\sqrt{\frac{f^{16}(x)}{f^{16}(x^2)} + 12x^2}$$





$$u = \frac{f(-x)}{\sqrt[3]{f(-x^3)}} \; ; \; v = \frac{f(-x^5)}{\sqrt[5]{f(-x^5)}}$$

75

$n = 2$ $\quad uv + \frac{x}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3$

$n = 3$ $\quad (uv)^3 + \left(\frac{x}{uv}\right)^3 = -\left\{\left(\frac{u}{v}\right)^6 - \left(\frac{v}{u}\right)^6\right\} - 9\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}$

$n = 4$ $\quad (uv)^4 + \left(\frac{x}{uv}\right)^4 = \left(\frac{u}{v}\right)^8 + \left(\frac{v}{u}\right)^8 - 8\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\} + 4\left(\frac{u}{v} + \frac{v}{u}\right)$.

$n = 5$ $\quad (uv)^5 + \left(\frac{x}{uv}\right)^5 + 5\left\{uv + \frac{x}{uv}\right\} + 15 = \left(\frac{u}{v}\right)^5 + \left(\frac{v}{u}\right)^5$.

$n = 7$ $\quad (uv)^7 + \left(\frac{x}{uv}\right)^7 = -\left\{\left(\frac{u}{v}\right)^7 - \left(\frac{v}{u}\right)^7\right\} - 7\left\{\left(\frac{u}{v}\right)^5 + \left(\frac{v}{u}\right)^5\right\}$

$\quad\quad - 7\left\{\left(\frac{u}{v}\right)^3 - \left(\frac{v}{u}\right)^3\right\} + 14\left(\frac{u}{v} + \frac{v}{u}\right)$.

$$u^2 = \frac{f(-x^2)}{\sqrt[3]{f(-x^3)}} \; ; \; v^2 = \frac{f(-x^5)}{\sqrt[5]{f(-x^5)}}$$

$n = 2$ $\quad uv + \frac{x}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 - 2\left\{\frac{u}{v} + \frac{v}{u}\right\}$

$n = 3$ $\quad (uv)^2 + \left(\frac{x}{uv}\right)^2 + 3\left(\frac{u}{v} + \frac{v}{u}\right)(uv + \frac{x}{uv})$

$\quad\quad = \left(\frac{u}{v}\right)^6 + \left(\frac{v}{u}\right)^6 - 6\left(\frac{u}{v} + \frac{v}{u}\right) - 9$.

$$u = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \cdots$$

$$v = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \cdots$$

$$u^3 v = 1 + 11x + u^6 v^6 + x^2 u v^{11}.$$

$$u = \frac{f(-x)}{x^{\frac{1}{24}} f(-x^5)} \qquad v = \frac{f(-x^5)}{x^{\frac{5}{24}} f(-x^{25})}$$

$n = 2$ $\quad (uv)^2 + \left(\frac{1}{uv}\right)^2 = \left(\frac{u}{v}\right)^3 - \left(\frac{v}{u}\right)^3$

$n = 3$ $\quad (uv)^3 + \left(\frac{1}{uv}\right)^3 + 5 = \left(\frac{u}{v}\right)^3 - \left(\frac{v}{u}\right)^3$.

$n = 4$ $\quad (uv)^4 + \left(\frac{1}{uv}\right)^4 = \left(\frac{u}{v}\right)^8 + \left(\frac{v}{u}\right)^8 - 5\left\{\frac{u}{v} + \frac{v}{u}\right\}$.

$n = 5$ $\quad (uv)^5 + \left(\frac{1}{uv}\right)^5 + 5\left\{uv + \frac{1}{uv}\right\} + 15 = \left(\frac{u}{v}\right)^5$

$n = 7$ $\quad (uv)^7 + \left(\frac{1}{uv}\right)^7 - 5 = \left(\frac{u}{v}\right)^7 - \left(\frac{v}{u}\right)^7 - 5\left\{\left(\frac{u}{v}\right)^5 + \left(\frac{v}{u}\right)^5\right\}$

$$u = \frac{f(-x^2)}{x^{\frac{1}{12}} f(-x^5)} \; ; \; v = \frac{f(-x)}{x^{\frac{5}{12}} f(-x^{25})}$$

$n = 2$ $\quad uv - \frac{5}{uv} = \left(\frac{u}{v}\right)^2 - \left(\frac{3v}{v}\right)^2$

$n = 3$ $\quad (uv)^3 - \left(\frac{5}{uv}\right)^3 = \left(\frac{u}{v}\right)^6 - \left(\frac{3v}{v}\right)^6 + \left(\frac{u}{v}\right)^2 - \left(\frac{3v}{u}\right)^2$

$\phi(a) = 1 + \frac{a x^{m+n}}{1-x} + \frac{a^2 x^{2m+2n}}{(1-x)(1-x^2)} + \cdots$

$\phi(a) = \phi(x) + a x^{m+n} \phi(a x^{2m})$; $\frac{\phi(a x)}{\phi(a)} = \psi(a)$

$1 + \psi(a) + a x^{m+n} \psi(a) \psi(a x^{2m}) \cdots \psi(a x^{2m(l-1)})$

$a = a, \; \beta = -\infty, \; m < \frac{1}{2}$ $\qquad$ $a \geqslant -\frac{(2m-1)^{2m-1}}{(2m)^{2m}}, \; m \geqslant \frac{1}{2}$

$1 + a\frac{x}{1-x} + \frac{a^2 x^4}{(1-x)(1-x^2)} + \cdots$

$= \prod_1^\infty \frac{1}{1 - a x^{2n-1}}$

$1 - x^n y_1 - x^{2n} y_2 - x^{3n} y_3 - \cdots$

$y_1 = \frac{1}{(1-x)\psi^6(x)}$, $y_2 = 0$, $y_4 = y_1 y_3$

$y_3 = \frac{x + x^3}{(1-x)(1-x^2)(1+x^3)\psi^6(x)} - \frac{x}{1-x} + \frac{3x^3}{1-x^3} + \frac{5x^5}{1-x^5} \cdots$ $\qquad$ $\frac{}{(1-x)^3 \psi^6(x)}$

If $x^6 = 1 + x^6$, then $0 = 1 - \frac{x}{1} + \frac{x^2}{1} - \frac{x^3}{1} + \frac{x^4}{1} - \frac{x^5}{1} + \frac{x^6}{1}$

If $x^n = 1$, $n \equiv 2, 3 \pmod 5$, (x primitive), then

$1 + \frac{x}{1} + \frac{x^2}{1} + \cdots + \frac{x^{n-1}}{1} = 0$.

If $x^n = 1$, $n \equiv 1, 4 \pmod 5$, (x primitive), then

$\frac{1}{1} + \frac{x}{1} + \frac{x^2}{1} + \cdots + \frac{x^{n-2}}{1} = 0$,

If $x^n = 1$, $n \equiv 0 \pmod 5$, $\left(\frac{x^2-1}{x^2-1} = 0\right)$, then

(i) $1 + \frac{x}{1} + \frac{x^2}{1} + \cdots + \frac{x^{n-1}}{1} = 0$; (ii) $\frac{1}{1} + \frac{x}{1} + \frac{x^2}{1} + \cdots \frac{x^{n-1}}{1} = 0$

If $\phi(a) = 1 + \frac{a x}{1-x} + \frac{a^2 x^3}{(1-x^2)(1-x^4)} + \frac{a^3 x^6}{(1-x^2)(1-x^4)(1-x^6)} + \cdots$

then $\phi(a)\phi(b) = 1 + \frac{x}{1-x}(a+b) + \frac{x^2}{(1-x^2)(1-x^4)}(a+b)(a+bx) + \frac{x^6}{(1-x^2)(1-x^4)(1-x^6)}(a+b)(a+bx)(a+bx^2) + \cdots$

$P_n(a) = 1 + a x \frac{1-x^n}{1-x} + a^2 x^4 \frac{(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)} + \cdots$

$P_{n-1}(a) P_{n-1}(ax) - P_n(a) P_{n-2}(ax) = (-a)^n x^{n(n-1)}$.

$P_n(a) = P_{n-1}(a) + a x P_{n-2}(ax)$

$P_n(a) = P_{n-1}(a) + a x^n P_{n-2}(ax)$ $\Bigg\}$ $P_0(a) = 1$, $P_{-1}(a) = 1$

$\frac{P_{n-1}(ax) + \varepsilon P_{n-2}(ax)}{P_n(a) + \varepsilon P_{n-1}(a)} = \frac{1}{1} + \frac{a x}{1} + \frac{a x^2}{1} + \cdots + \frac{a x^n}{1+\varepsilon}$

If $\varepsilon = \frac{1}{1} + \frac{a x}{1} + \frac{a x^2}{1} + \cdots + \frac{a x^n}{1+\lambda\varepsilon}$

then $\varepsilon = \frac{P_{n-2}(ax)}{P_{n-1}(a)} + z$, where

$\lambda z^2 + z\{P_n(a) + \lambda P_{n-2}(ax)\} = (-a)^n x^{n(n-1)}$

If $x^n = 1$ (x primitive), then $P_{n-1}(a) + a P_{n-3}(ax) = 1$

$P_{n-2}(a) P(ax) - P_{n-3}(ax) P(a) = (-a)^{n-1} x^{n(n-1)} P(ax^n)$

If $x^n = 1$, (x primitive), then

	$P_{n-2}(1)$,	$P_{n-2}(x)$	$P_{n-1}(1)$	$P_{n-2}(x)$
$n \equiv 1, 4 \pmod 5$	$+\frac{5}{x}$	$+\frac{5}{x}$	1	0
$n \equiv 2, 3 \pmod 5$	$-\frac{5}{x}$	$-\frac{5}{x}$	0	1
$n \equiv 0 \pmod 5$	0 *	0 *	$-(x^{-2}+x^2)$	$-(x^{-1}+x)$

* x need not be primitive; it is enough that $x^5 = 1$

The analytic property of ε in relation to a (as in Jacobi's Fundamenta)

$$1 + q(a_1 - 1) + q^2 a_2 + q^3(a_3 + 1) + q^4(a_4 + a_2 + 1)$$
$$+ q^5(a_5 + a_3 + a_1 + 1) + q^6(a_6 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$+ q^7(a_3 + 1)(a_4 + a_2 + 1) + q^8 a_2(a_6 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$+ q^9 a_2(a_3 + 1)(a_4 + a_2 + 1) + q^{10} a_2(a_3 + 1)(a_5 + a_3 + a_1 + 1)$$
$$+ q^{11} a_1 a_2(a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + 2)$$
$$+ q^{12}(a_3 + a_2 + a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$\times(a_4 - 2a_3 + 2a_2 - a_1 + 1).$$
$$+ q^{13}(a_1 - 1)(a_2 - a_1 + 1)(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6$$
$$+ 4a_5 + 6a_4 + 8a_3 + 9a_2$$
$$+ 9a_1 + 9)$$
$$+ q^{14}(a_2 + 1)(a_3 + 1)(a_4 + a_2 + 1)(a_5 - a_3 + a_1 + 1)$$
$$+ q^{15} a_1 a_2(a_5 + a_4 + a_3 + a_2 + a_1 + 1)(a_7 - a_6 + a_4 + a_1)$$
$$+ q^{16}(a_3 + 1)(a_3 + a_2 + a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$\times(a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3)$$
$$+ q^{17}(a_2 + 1)(a_3 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$\times(a_7 - a_6 + a_3 + a_1 - 1)$$
$$+ q^{18}(a_4 + a_2 + 1)(a_5 + a_2 + a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$\times(a_6 - 2a_5 + a_4 + a_3 - a_2 + 1)$$
$$+ q^{19} a_2(a_1 - 1)(a_5 + a_2 + 1)(a_5 + a_2 + a_1 + 1)$$
$$\times(a_9 - a_7 + a_4 + 2a_3 + a_2 - 1)$$
$$+ q^{20}(a_2 - a_1 + 1)(a_5 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$$
$$\times(a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3)$$
$$+ q^{21} a_1 a_2(a_5 + 1)(a_2 - a_1 + 1)(a_5 + a_4 + a_2 + a_1 + 1)$$
$$\times(a_8 - a_6 + a_4 + a_1 + 2).$$

13. $(a_1-1)(a_2-a_3+1)$ 12. $(a_2+1)(a_3+1)(a_4+a_2+1)$

16. $a_1 a_2^2 (a_5+a_4+a_3+a_2+a_1+1)$

16. $(a_3+1)(a_3+a_4+a_1+1)(a_1+a_4+a_3+a_2+a_1+1)$

17. $(a_2+1)(a_3+1)(a_3+a_4+a_3+a_2+a_1+1)$

18. $(a_4+a_2+1)(a_3+a_4+a_1+1)(a_3+a_4+a_3+a_2+a_1+1)$

19. $a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1)$

20. $(a_3+1)(a_2-a_1+1)(a_3+a_4+a_3+a_2+a_1+1)$

21. $a_1 a_2(a_3+1)(a_2-a_1+1)(a_3+a_4+a_3+a_2+a_1+1)$

22. $a_2(a_3+1)(a_1-1)$

23. $(a_1-1)(a_4+a_2+1)$

24. $(a_3+1)(a_4+a_2+1)(a_3+a_2+a_1+1)$

25. $a_2(a_1-1)(a_3+a_4+a_3+a_2+a_1+1)$

26. $a_2(a_3+1)(a_3+a_2+a_1+1)$

2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28
30, 31, 34, 40, 42, 45, 46, 47, 50, 55, 57,
58, 59, 62, 66, 70, 74, 76, 78, 79, 86, 94,
98, 106, 110, 122, 126, 130, 142, 154, 158, 170, 174,
206

14, 16, 18, 24, 32, 48, 56, 72, 82, 88, 90, 104, 114
138, 146, 162, 178, 186, 196, 202, 210, 218, 226, 234, 242
250, 266

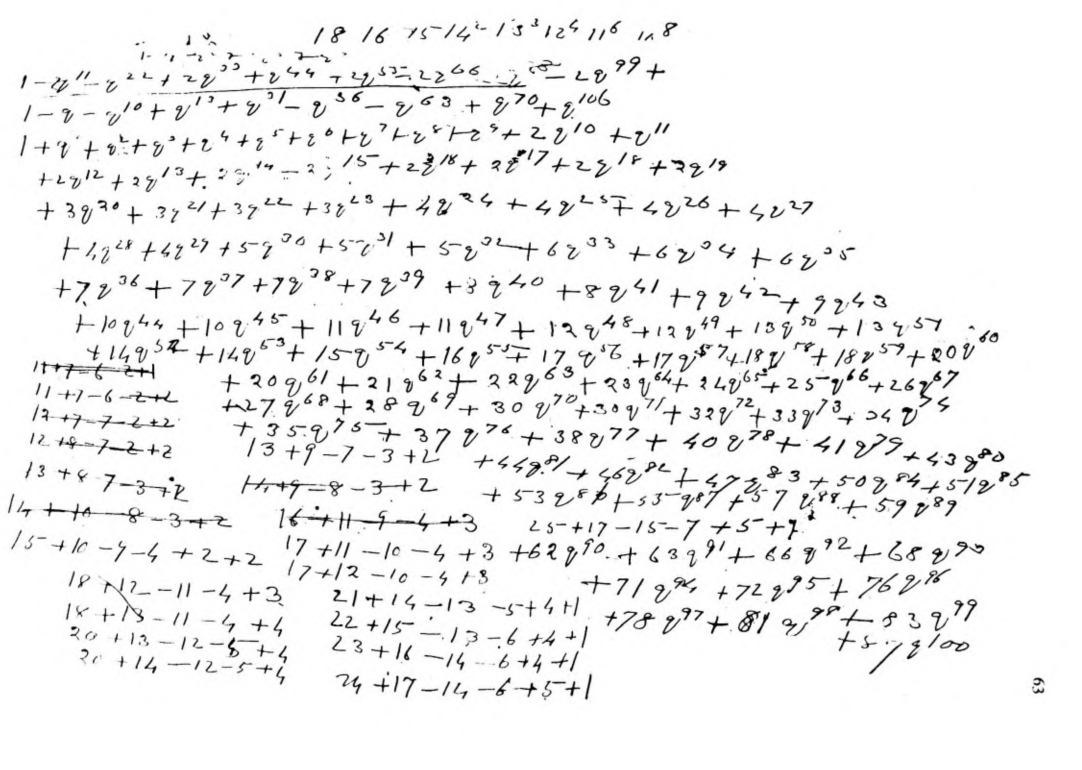
46 12 20 36 38 44 52 56 60 68 76 92 100
118 134 150 166 182 190 222 238 254
270 286 302,

1 7 17 23 33, 39, 49, 63, 71, 73, 81, 87, 89, 95
105 111 119 121 127 143 157

3, 5, 13, 29, 35, 37, 43, 53, 61, 67, 69, 77, 83, 85, 91, 93
99, 107

+ 1 — 12, 18, 24, 30, 42, 54,
+ 1 — 7, 31, 43, 49, 67, 73, 85,
91, 97, 109
— 1 — 2, 20, 26, 32, 38, 56, 62, 68
— 1 — 3, 9, 15, 21, 27, 39
— 1 — 4, 10, 28, 40, 52, 64, 70, 130
76, 94, 106, 118,
+ 1 — 5, 11, 29, 35, 41, 47, 53, 59
71,

0 — 1, 6, 13, 19, 25, 37
55, 61
8, 14, 44, 33
22, 34, 46, 58, 82
14, 21,
3, 39



$21\ 19\ 18\ 17^2\ 16^2\ 15^4\ 14^4\ 13^2\ 12$

$66\,32\,10^2$

$7\,6\,3\,2\,10$

$15\ 11\ 10\ 9\ 8\ 7^3$

$8-7+6-5+4+3+2+1+0$

$0+1^2+2^3-3^2+4^2-5+6$

$7-6-5+4^2\quad 8\ 7^2$

$1+0\qquad -13-11-9-8$

$15\ 13\ 11\ 10\ 9\ 8\ 7$

$8-6+4+1+0^2\qquad 10$

$11+10+7$

$21\ 19\ 18\ 17\ 16\ 15^{-2}$

$17\qquad 15-14\ 13$

$16\qquad\quad 15-14\ 13$

$15-\quad 14\ 13$

$\quad\quad 14$

32

$6\ 3\ 0^2$

$6\,3\,2\,1$

$3\ 1$

$1-2b+b^2+2b^3+b^4-b^5+2b^6$

$8,7^2$

$1-2b+b^2+2b^3+b^4-b^5+2b^6$

$+3b^7+2b^8+7b^9+2b^{10}+2b^{11}+b^{12}+2b^{13}+b^{14}$

$+2b^{15}+b^{16}+2b^{17}+2b^{18}+7b^{19}+2b^{20}+2b^{21}+3b^{22}+2b^{23}+2b^{24}+b^{25}+2b^{26}$

$202\ 224\ 246\ 268\ 301\ 323\ 345$

$6+4-7+1$

$7+5-2-1$

$1-2-7+4\ 1$

$1-1+5+8$

$1-1-4+5\ 1-2-5+9$

$1-1-4+5$

$1-1-4+3$

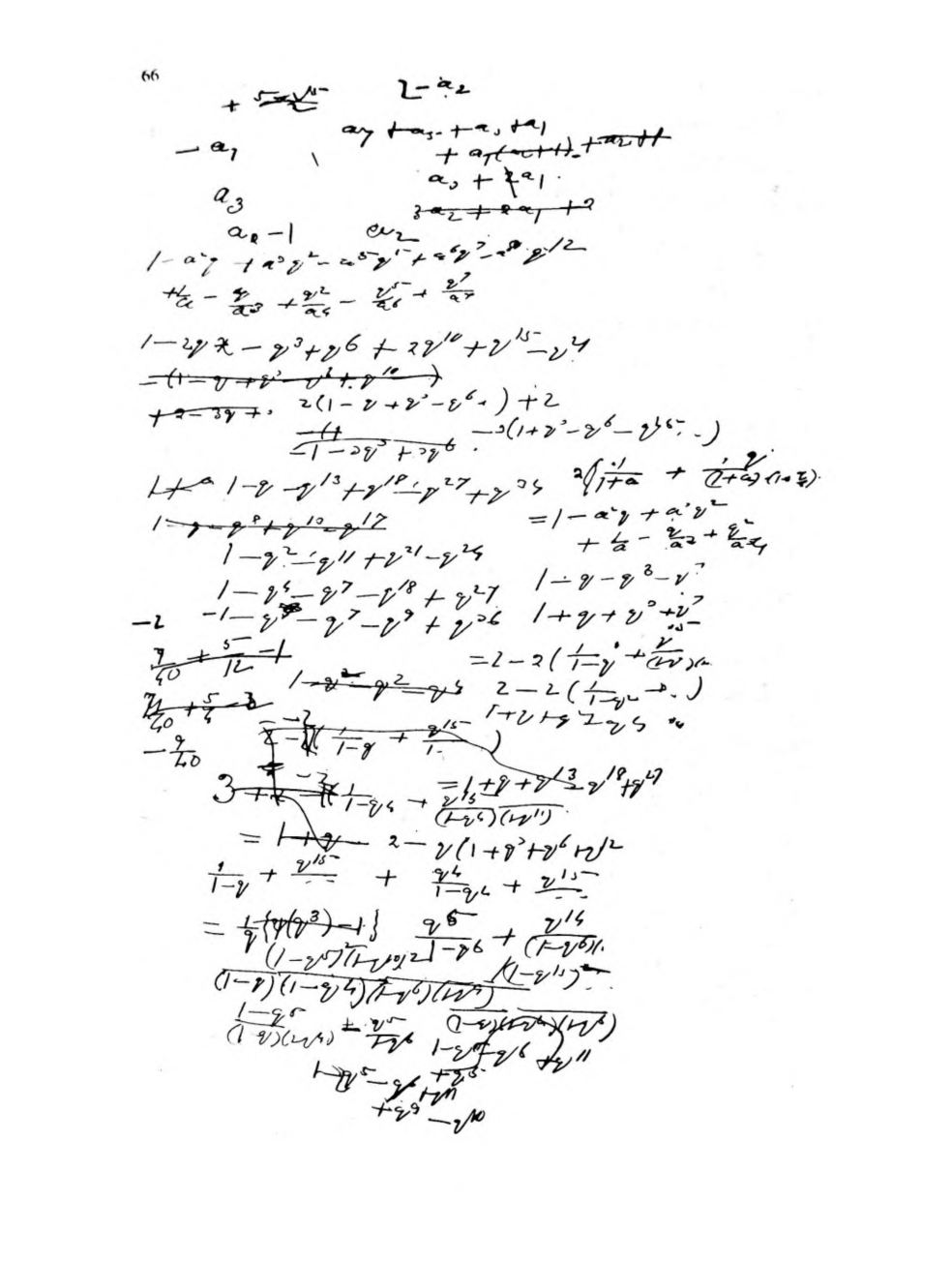
$1+4+4+4$

$1+1+4+4$

$1+3+5$

$1-1+4$

$1+3+4$



$$\frac{(\lambda+1)(\lambda+1)}{(\lambda^2+1)(\lambda^2+1)(\lambda^2+1)}$$

$$\frac{1+\lambda}{\lambda^2+1}$$

$$\frac{(2+\lambda)(\lambda+1)}{(\lambda^2-\lambda+1)\lambda^2} + \frac{(\lambda-1)(\lambda+1)}{\lambda^2} + \lambda^{-1} + \lambda^{-1} -$$

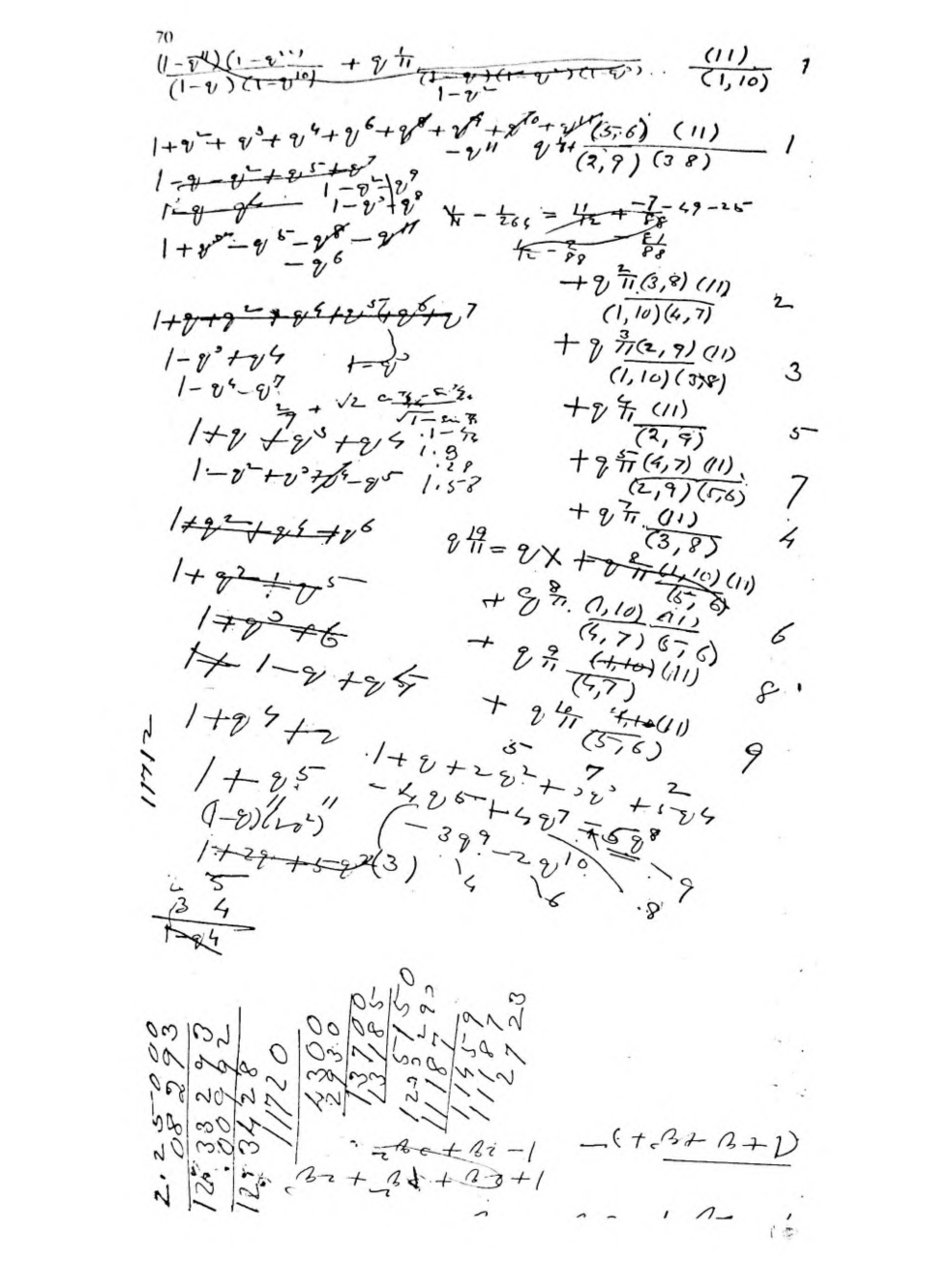
$$\frac{(\lambda-1)(\lambda^2+1)(\lambda+1)}{(\lambda^2+\lambda+\lambda^2-\lambda-1)\lambda^2} +$$

$$\frac{(\lambda^2+1)}{(\lambda^2+\lambda^2-2)\lambda^2} +$$

$$\frac{\lambda^2+1}{(\lambda^2+\lambda^2+1)\lambda}$$

$$\lambda^2 - \lambda^2 - \lambda^2 - \

$$
\begin{array}{llllll}
1 & +v^{12} & +v^{24} & +v^{36} & +v^{48} & +2v^{60} \\
+q & +v^{13} & +v^{25} & +2v^{37} & +2v^{49} & +2v^{61} \\
+v^{2} & +v^{14} & +v^{26} & +v^{38} & +v^{50} & +v^{62} \\
+0 & -v^{15} & +0 & -v^{39} & -v^{51} & -v^{63} \\
+0 & +0 & -v^{28} & -v^{40} & -v^{52} & \\
+0 & +0 & -v^{29} & +0 & -v^{53} & \\
+v^{4} & +0 & +v^{30} & +v^{42} & +v^{54} & \\
+v^{7} & +v^{19} & +v^{31} & +v^{43} & +2v^{55} & \\
+0 & +0 & +0 & +0 & +v^{56} & \\
+0 & 0 & -v^{33} & -v^{45} & -v^{57} & \\
-v^{10} & -v^{22} & -v^{34} & -2v^{46} & -2v^{58} & \\
+0 & +0 & +0 & +0 & +0 & \\
 & & & & -v^{59} &
\end{array}
$$



0. (2,4) (1,6) (2,5) 1

1. (7) (1,6) 1

2. (2,5) (7) (6,6) (3,4) 6

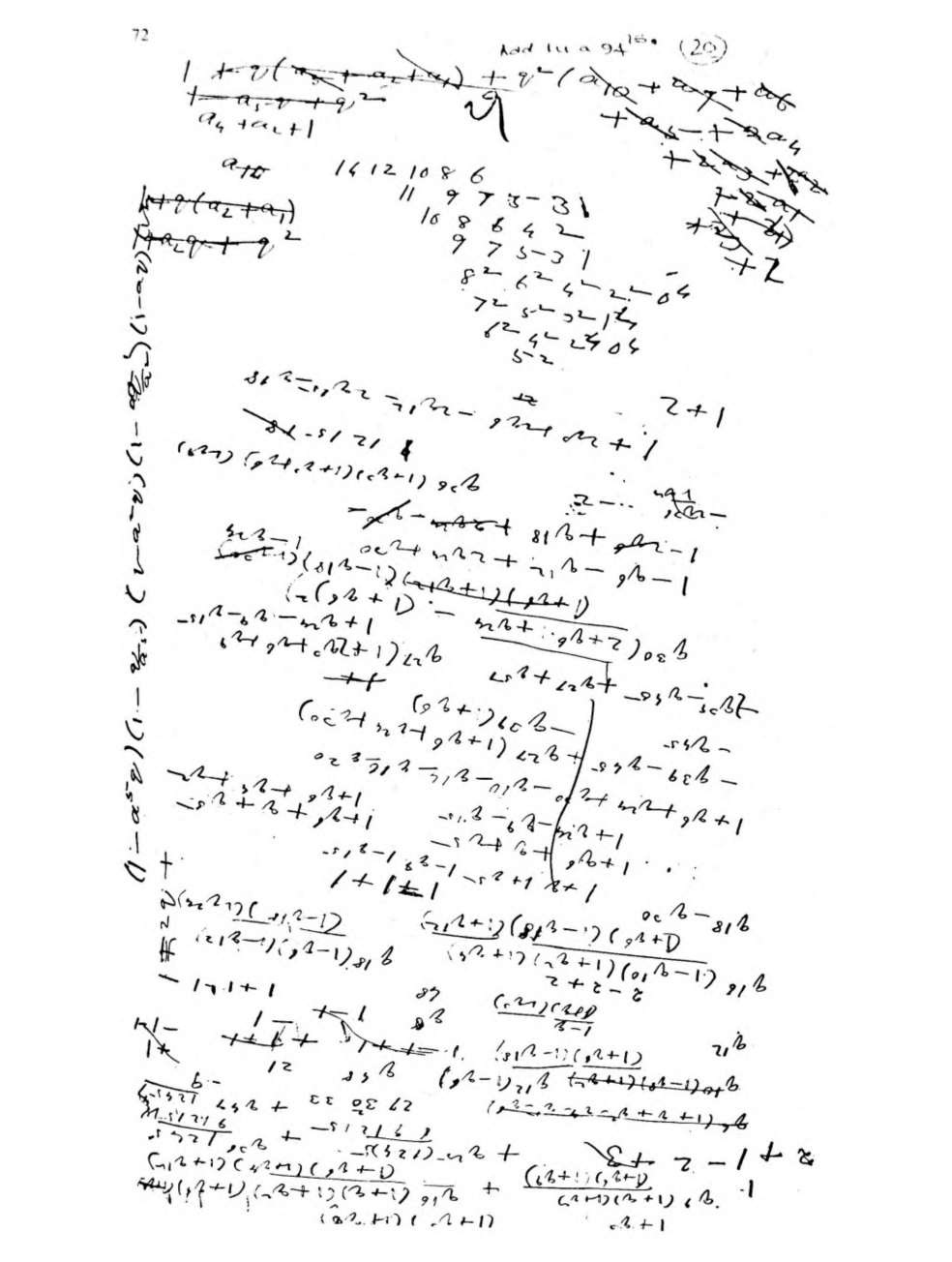
3. (7) (2,5) 3

4. (7) (3,4) 5

5. (1,6) (3,4) (7,5) 4

6. (3,4) (2,5) (1,6) 6

22



Addition 94 (35)

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1	(2, 9)	$1+9^{3}-9^{5}$
2	(1, 10)	$1-9^{10}+9^{7}(1-2^{7}-9^{3}$
3	(3, 8)	$1-9^{5}/8+9^{2}(1-9^{2}$
4	(4, 7)	$\frac{8\cdot 1}{2\cdot 8}$
5	(5, 6)	$1+9^{2}$
6	(5, 6)	$\frac{2\cdot 8}{5}$
7	(5, 6)	
8	(4, 7)	
9	(3, 8)	$1+2-2^{3}$
10	(1, 10)	$\frac{1-2^{7}-2^{8}}{1-2^{5}}+2\cdot(1-2^{5}-2^{10})$

$$(1-2^{7}-2^{8}+2^{24}+2^{31})\,(1-2^{5}-2^{10}+2^{17}+2^{34})$$

$$1-2^{5}+2^{7}+2^{15}-2^{18}-2^{27}+2^{34}+2^{48}$$

$$\frac{1-2^{2}}{1-2^{5}}\cdot 2^{3}(1-2^{5})$$

$$2-3\cdot 2^{5}+2^{10}-2^{15}+2^{25}-2^{30}$$

$$1-2^{5}-2^{6}+2^{11}+2^{12}-2^{17}-2^{18}+2^{23}+2^{24}-2^{29}-2^{30}$$

$$1+2^{5}-2^{6}-2^{11}+2^{12}+2^{17}-2^{18}-2^{23}+2^{24}+2^{29}-2^{30}$$

$$\frac{1-2^{5}}{1-2^{2}}$$

$$\frac{1+2^{5}}{1+2^{2}}$$

$$\frac{1-2^{5}}{1-2^{2}}$$

$$\frac{1-2^{5}}{1-2^{2}}$$

$$\frac{1-2^{5}}{1-

$$\frac{(\beta-1)(5\beta-1)(\beta-1)}{\beta^3+\beta^2+9\beta+5\beta^2+5\beta^2+\beta+1}$$

$$\beta^4+\beta^3+\beta^2+9\beta+5\beta^2+5\beta^2+\beta+1=$$

$$\left\{\cdot+\frac{\beta+1}{\beta}+1\right\}+\frac{\beta+1}{\beta^2+9\beta+1}$$

$$\frac{(\beta-1)(-\beta-1)}{01\beta}+\frac{-\beta-1}{1}\Big)^2$$

$$\frac{(\beta-1)(-\beta-1)}{(\beta-1)}$$

$$\frac{\beta^2+9\beta+\beta-1}{(\beta-1)(-\beta-1)}$$

$$\frac{-\beta-1}{\beta}$$

$$\frac{(\beta+1)(-\beta-1)(\beta-1)}{-\beta-1}$$

$$\frac{\beta+1}{-1\beta+1}$$

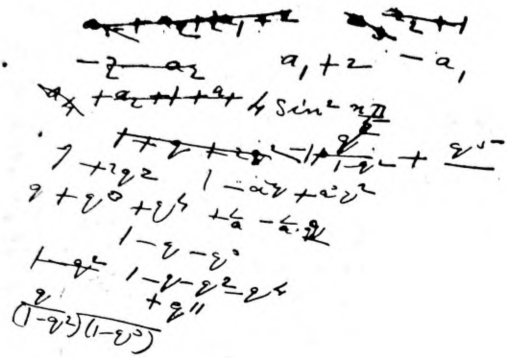
$$\frac{-\beta-1}{\beta}$$

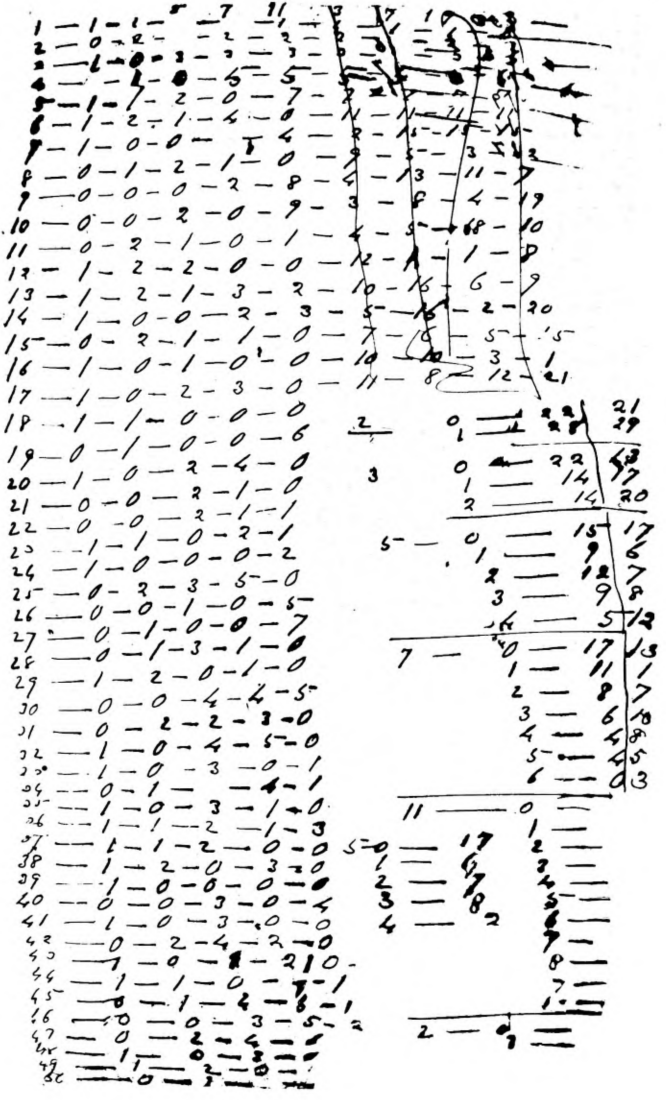
$$\frac{\beta^2+\beta+9\beta+\beta+1}{\beta^2+\beta+1}$$

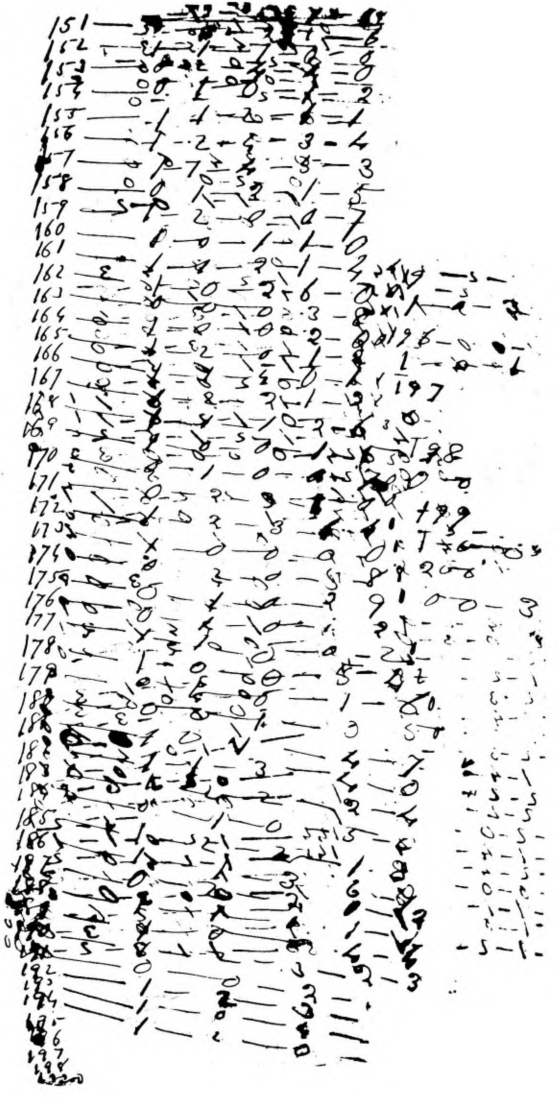
$$\frac{\beta^2+\beta+1}{\beta^2+\beta+1}$$

$$\frac{\beta^2+9\beta+\beta+1}{\beta^2+\beta+1}$$

$$\frac{\beta^2







1357 37

$$e^{\lambda} = 10^6 \cdot 10^{\varepsilon}$$

$$\lambda = 13.82 + 10^6 \log \lambda$$

$$= 10^6 \log \lambda \,(1+\varepsilon) \qquad \varepsilon = \frac{13.82}{10^6 \log \lambda}$$

Now

$$\log \lambda = 13.82 + \log\log\lambda + \varepsilon' \quad (0 < \varepsilon' < \varepsilon)$$

$$\lambda = 10^6 \left\{ 13.82 + \log\log\lambda + \varepsilon' \right\} (1+\varepsilon)$$

$$\log\log\lambda = \log 13.82 + \log\left(1 + \frac{\log\log\lambda + \varepsilon'}{13.82}\right)$$

Now $\log\lambda < 20 < e^3$

$$\log\log\lambda < 3$$

$$\frac{\log\log\lambda + \varepsilon'}{13.82} < \frac{1}{4}$$

$$\log\log\lambda = 2.63 + \eta \qquad \eta < \frac{1}{4}$$

$$\lambda = 10^6 \left[13.82 + 2.63 + \eta' \right] \left[1+\varepsilon \right] \quad (\eta' < \tfrac{1}{4})$$

$$= 10^6 \left[16.45 + \eta'' \right] \qquad \eta'' < \frac{1}{4}$$

Letters from Srinivasa Ramanujan to G.H. Hardy and some sheets from the Lost Notebook

(1)5) Trinity House
Dear Mr Hardy,

you told me that you can prove at present that all numbers can be expressed as the sum of 33 fourth powers. I wonder whether you can get a better result by starting with the following function instead of $1 + 2x + 2x^{16} + 2x^{81} + \dots$.

We have $\left(= \frac{x}{1-x^2} + \dots\right)$

$$\sum_{-\infty}^{\infty} x^{\mu^2+\mu\nu+\nu^2} = 1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^3} + \frac{x^4}{1-x^5} \dots\right)$$

It follows that if $\phi(x) = x + x^{2^2} + x^{3^2} + x^{4^2} + \dots$ then

$$\sum_{-\infty}^{\infty} x^{(\mu^2+\mu\nu+\nu^2)^2}$$

$$= 1 + 6\{\phi(x) - \phi(x^4) + \phi(x^{16}) - \phi(x^{25}) + \phi(x^{49}) - \phi(x^{64}) + \dots\}$$

where 1, 4, 16, 25 are the squares of natural numbers without the multiples of 3. Suppose now that you can prove that for a certain value

then but for the cosines

$t_0 = 1$, $t_1 = 0$, $t_2 = \left(\frac{1}{3}\right)^{\frac{1}{2}s}$,

$t_4 = \left(\frac{1}{2}\right)^{\frac{1}{2}s}$, $t_6 = 0$, $t_{12} = \left(\frac{1}{8}\right)^{\frac{1}{2}s}$

and so there is a slight increase in some of the T's over the t's, which can naturally be expected. Now in the original problem the value of s is 5 in order that the coefficients are positive using the partition method (though 4 is the right number). This being so the value of s in the present problem must certainly be greater than 5 due to the slight increase in the value of T's (using the partition method). If it is possible to use your arguments to this

all the coefficients after a certain point in the expansion of

$$\left\{1 + 6\left(\phi(x) - \phi(x^{\frac{1}{2}}) + \phi(x^{\frac{1}{4}}) - \phi(x^{\frac{1}{8}}) + \cdots\right)\right\}^s \quad (2).$$

(1) are all greater than 0 by using the partition method. Then since

$$\mu^4 + \nu^4 + (\mu+\nu)^4 = 2(\mu^2 + \mu\nu + \nu^2)^2$$

it follows that all even numbers can be expressed as the sum of 3s fourth powers after a certain point, & since any odd number is the sum of a fourth power and an even number it follows that all odd numbers can be expressed as the sum of $3s+1$ fourth powers after a certain point onwards

(2) Now the asymptotic value of the coefficient of x^n in (1) according to the partition method

is

$$\frac{C^s n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)}\left\{T_1 + T_2 + T_3 + T_4 + \cdots\right\}$$

where T's are the terms arising out of the partition method; e.g.,

$$T_1 = 1; \quad T_2 = (-1)^{n+s}\left(\tfrac{1}{2}\right)^s \text{ or } \cdots$$

$$T_3 = 2\cos\left(\frac{\pi s}{2} - \frac{2\pi n}{3}\right).\left(\frac{2}{3}\right)^{\frac{1}{2}s} \text{ or } \cdots$$

$$T_4 = 2\cos\left(\tfrac{1}{2}\tan^{-1}3 - \frac{\pi n}{2}\right).\left(\frac{5}{9}\right)^{\frac{1}{4}s} \text{ or } \cdots$$

$$T_6 = 2\cdot\cos\left(\frac{\pi s}{2} - \frac{\pi n}{3}\right).\left(\frac{1}{12}\right)^{\frac{1}{2}s} \text{ or } \cdots$$

$$T_{12} = \left\{\frac{2}{2}\cos\left(\tfrac{1}{2}\tan^{-1}\tfrac{1}{3} - \cdots\right) + (-1)^s \cdot 2\cos\left(\tfrac{1}{2}\tan^{-1}\tfrac{1}{3} - \frac{5\pi n}{6}\right)\right\}\left(\frac{1}{48}\right)^{\frac{1}{2}s}.$$

If the constant $C = 3\sqrt{\pi}\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} + \cdots\right)$

$$= \pi\sqrt{\left(\tfrac{1}{2}\pi\right)}.$$

If the asymptotic value of the coefficients of x^n in $(1+x+x^2+x^3+x^4+\cdots)^s$ be

$$\frac{\pi^{\frac{1}{2}s} n^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}s)}\left\{t_1 + t_2 + t_3 + t_4 + \cdots\right\}$$

function I don't think the value of s will be so high as 10.

$s = 10$ gives { even nos. - 30 fourth powers
odd nos. — 31 fourth powers

$s = 5$ is certainly impossible, for if we had $s = 5$ then all even numbers after a certain point would be expressible as the sum of 15 fourth powers which is certainly false. As a matter of fact it appears that all even numbers without exception can be expressed as the sum of 18 fourth powers while the odd numbers without exception can be expressed as the sum of 19 fourth powers. There can be a better result than this (for all numbers) since $78 = 4 \cdot 2^4 + 14 \cdot 1^4$ and $79 = 4 \cdot 2^4 + 15 \cdot 1^4$ though it may be likely that all numbers after a certain point can be expressed as the sum of 16 fourth powers

From the formula

$$(\mu^4+\nu^4+(\mu+\nu)^4 = 2(\mu^2+\mu\nu+\nu^2)^2$$

it is quite easy to prove that the number of ways in which a number N can be expressed as the sum of 3 fourth powers is greater than

$$> 2^{(\frac{1}{2}-\varepsilon)\frac{\log N}{\log\log N}}$$

for an infinity of values of N.

If $a_\lambda = 1$, $\lambda = \mu^2+\nu^2$

and $a_\lambda = 0$, $\lambda \neq \mu^2+\nu^2$

then $a_1 + a_2 + \cdots + a_n \sim \frac{cn}{\sqrt{\log n}}$

where c is a constant. ~~c is known~~

Could you tell me if there are results like this to other powers than 4 to say if $a_\lambda = 1$, $\lambda = \mu^s+\nu^s$

and $a_\lambda \neq 0$, $\lambda \neq \mu^s+\nu^s$

then is the order of $a_1+a_2+\cdots+a_n$ known?

Yours

S. Ramanujan

(1) Mal Lock
Sunday

Dear Mr Hardy,

In one of my letters I wrote about the least number of terms which will give the nearest integer to the actual coefficient in 1/3 problem. It will be extremely difficult to prove such a result. But we can prove this much as follows.

$$\sum \alpha_n x^n = \frac{1}{1-x}\left(1 + \frac{x^2}{1-x^2} + \frac{x^{3^2}}{1-x^{3^2}} + \cdots\right)$$

(1) $$\alpha_n = c\left\{e^{\frac{2n\pi}{3}} + (-1)^n \frac{e^{\frac{n\pi}{2}}}{2^{\frac{1}{2}}} + 2\cos\left(\frac{2n\pi}{5} + \theta(2^2 \cdot 1)\right)\frac{e^{\frac{2n\pi}{5^{\frac{1}{2}}}}}{5^{\frac{1}{4}}} + \cdots\right.$$
$$\left. + \left\{2\cos\left(\frac{2n\pi}{65^{\frac{1}{2}}} + \theta(2^2 \cdot 1 \cdots)\right) + 2\cos\left(\frac{2n\pi}{65^{\frac{1}{2}}} + \theta(2^2 \cdot 1 \cdots)\right)\right\}\frac{e^{\frac{2n\pi}{65^{\frac{1}{2}}}}}{65^{\frac{1}{4}}} + \cdots\right\}$$

where $c = \frac{2\sqrt{2}}{9}\left\{\frac{\Gamma(\frac{1}{4})}{\pi^{\frac{1}{2}}}\right\}^{16} = .94373\cdots$

We shall first prove that, if we take only

(3) $$\left[\frac{\frac{3}{2}n(1-\epsilon)}{(\log n)^{\frac{1}{2}}\sqrt{(1-\frac{1}{5^2})(1-\frac{1}{17^2})(1-\frac{1}{17^2})\cdots}}\right]$$

terms in the right hand side of (1), ϵ being any positive number less than 1 and 5, 13, 17, ... are primes of the form $4k+1$, then it is possible to find an infinity of values of n for which α_n is not the nearest integer to the sum of the asymptotic series up to (3) terms. Suppose that $\varpi(\lambda)$ is the number of sums of any two squares such as 1, 2, 4, 5, 8, 9, 10, 13, 16, ... not exceeding λ and that $p(\lambda)$ is the number of sums of two squares that are prime to one another, such as 1, 2, 5, 10, 13, 17, 25, ... not exceeding λ. Then it is easy to see that

(2)

(3) $p(\lambda) = \Delta(\lambda)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots$

where 2, 3, 7 ... are the primes excluding those of the form $4k+1$ and the right hand ... the Δ is understood as $\Delta(\lambda) - \Delta\left(\frac{\lambda}{3}\right) - \Delta\left(\frac{\lambda}{7}\right) + \Delta\left(\frac{\lambda}{21}\right) - \cdots$

... has proved that

(4) $\Delta(\lambda) \sim \frac{\lambda}{\log\lambda}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{(1-\frac{1}{11^2})\cdots}}$

3, 7, 11 ... being the primes of the form $4k+3$.

* Please see if this constant is correct. I ... somewhat ... note that

$$\left(1+\tfrac{1}{7}\right)\left(1+\tfrac{1}{11}\right)\left(1+\tfrac{1}{19}\right) = \sqrt{2\left(1-\tfrac{1}{3^2}\right)\left(1-\tfrac{1}{7^2}\right)\left(1-\tfrac{1}{11^2}\right)\left(1-\tfrac{1}{19^2}\right)}$$

(the first 4 terms in the ...).

We can easily prove from (3) ... that

$$p(\lambda) \sim \frac{\lambda}{\log\lambda}\cdot\frac{1}{\sqrt{2\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots}}\cdots;$$

that is to say

(5) $$p(\lambda) \sim \frac{3\lambda}{2\pi\log\lambda}\cdot\frac{1}{\sqrt{\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\cdots}}$$

5, 13, 17 being the primes of the form $4k+1$.

Suppose we take only ... and the ... the L... is of the form

$$C\left(2\cos\cdots + 2\cos\cdots + \right)\frac{e^{\frac{2n\pi}{\lambda}}}{\lambda^{\frac{1}{2}}}$$

where λ is a number belonging ... to

1, 2, 5, 10, 13 ...

and C is the same as in (1) Then obviously

$$p(\lambda) = (2)$$

It follows from (2) and (5) that

(6) $$\lambda \sim \frac{\pi}{2}\frac{n}{\log n}(1-\varepsilon)$$

(3) (P)

Now choose a prime p of the form $4k+1$ such that $p > \lambda$ and $p \sim \lambda$.

This is quite possible since consecutive primes of the form $4k+1$ are asymptotically equivalent to one another just as ordinary primes.

Consider the asymptotic series for $\alpha_n, \alpha_{n-1}, \alpha_{n-2} \cdots \alpha_{n-p+1}$ remembering that we take only

$$\left[\frac{\frac{3}{4}n(1-\varepsilon)}{(\frac{1}{4}n)^{\frac{3}{2}}\cdot\bar{\varepsilon}(1-\frac{1}{4}\bar{\varepsilon})(1-\frac{1}{4}\bar{\varepsilon}^2)\cdots}\right], \left[\frac{\frac{3}{4}(n-1)(1-\varepsilon)}{\{\frac{1}{4}(n-1)\}^{\frac{3}{2}}\sqrt{(1-\frac{1}{4}\bar{\varepsilon})(1-\frac{1}{4}\bar{\varepsilon}^2)}\cdots}\right], \cdots$$

terms in $\alpha_n, \alpha_{n-1}, \cdots$ respectively.

A term $C \cdot 2\cos\left(2\frac{n\pi}{p} + \theta\tan^{-1}\theta\right)\frac{e^{\frac{2n\pi}{p}\tau}}{p^{\frac{3}{2}}}$

will occur in α_n beyond the terms we have taken. Similarly

$$C \cdot 2\cos\left(2\frac{(n-1)\pi}{p} + \theta\tan^{-1}\theta\right)\frac{e^{\frac{2(n-1)\pi}{p}\tau}}{p^{\frac{3}{2}}}$$

will occur in α_{n-1} beyond the terms we have taken. And so on. In all these θ and p remain the same throughout. There is only one term $2\cos$ instead of many $\cos$'s since p can be expressed as the sum of 2 Squares in one and only way. Now take a number K such that

$$0 < K < 2.$$

Then one of the p numbers

$$2\cos\left(\frac{2n\pi}{p} + \theta\tan^{-1}\theta\right), 2\cos\left(\frac{2(n-1)\pi}{p} + \theta\tan^{-1}\theta\right), \cdots 2\cos\left(\frac{2(n-p+1)\pi}{p} + \theta\tan^{-1}\theta\right)$$

we can certainly find as numerically

$$2\cos\left(\frac{2(n-s)\pi}{p} + \beta\tan^{-1}\theta\right) > K, \quad (4)$$

and $2\cos 2(n-\ldots)\pi - \ldots) < -K.$

The corresponding term in α_{n-s} is greater than $KC\,e^{\frac{2(n-s)}{p}\pi}$ and that in α_{n-t} is less than $-KC\cdot e^{\frac{2(n-t)}{p}\pi}$. Case not-negative and

Now we know that s and t do not exceed a and $p \sim \frac{\pi n}{2\log n}(1-\varepsilon)$. Hence

$$KC\,\frac{e^{2(n-s)\pi}}{p} = KC\,\frac{16(\log n)^4}{\pi^4 n^4}\,e^{4\frac{\log n}{1-\varepsilon}\{1+o(1)\}}$$

$$= KC\frac{16}{\pi^4}(\log n)^4 n^{4\frac{\varepsilon}{1-\varepsilon}} + o(1)$$

Precisely in the same way

$$-KC\,\ldots = -KC\frac{16}{\pi^4}(\log n)^4 n^{4\frac{\varepsilon}{1-\varepsilon}} + o(1)$$

The right-hand side surpass all limits as $n \to \infty$.

Thus we see that the difference between α_n and the sum of the series for α_n to (2) terms is not even for $n = \infty$ as $n \to \infty$.

We shall now prove that, if ε is any positive number and if we take

$$(7) \qquad \left[\frac{n(1+\varepsilon)}{(\log n)^{\frac{3}{2}}\sqrt{(1-\frac{1}{3-\varepsilon})(1-\frac{1}{7-\varepsilon})\ldots}}\right]$$

terms, then the sum of (7) terms will be α_n for all sufficiently

(5-)

large values of n. As before let the $(\lambda+1)^{th}$ term be

(8) $\quad C\left\{2\cos\cdots + 2\cos\cdots + \cdots\right) \frac{e^{\frac{2\pi n}{\lambda}}}{\lambda^z}$

where λ is a number belonging to the class
1, 2, 5, 10, 13, 17, ...
and let the subsequent numbers be $\lambda', \lambda'', \lambda''', \ldots$
Now the number of cos's in (8) is the number of ways in which λ can be expressed as the sum of two squares, etc. It follows that in (8)

$$\left|2\cos\cdots + 2\cos\cdots + \cdots\right| \leq d(\lambda)$$

where $d(\lambda)$ is the number of divisors of λ. Hence the absolute value of the sum of all the remaining terms is less than

$$C\left\{\frac{d(\lambda')}{\lambda'^{\frac{z}{2}}} e^{\frac{2\pi n}{\lambda'}} + \frac{d(\lambda'')}{\lambda''^{\frac{z}{2}}} e^{\frac{2\pi n}{\lambda''}} + \cdots\right.$$

$$< C\left\{d(\lambda+1) = O\left\{\lambda'^{-4+o(1)} e^{\frac{2\pi n}{\lambda'}} + \lambda''^{-4+o(1)} e^{\frac{2\pi n}{\lambda''}} + \cdots\right.\right.$$

$$= O\, e^{\frac{2\pi n}{\lambda}} \left(\lambda'^{-4+o(1)} + \lambda''^{-4+o(1)} + \cdots\right\}$$

$$= O\left\{e^{\frac{2\pi n}{\lambda}} \left((\lambda+1)^{-4+o(1)} + (\lambda+2)^{-4+o(1)} + \cdots\right)\right.$$

$$= O\,\lambda^{-3+o(1)}\, e^{\frac{2\pi n}{\lambda}}$$

But $p(\lambda) = (7)$ so that, in view of (5),

(9) $\quad \lambda \sim \frac{2\pi n}{3\log n}(1+\epsilon)$

Hence $O\left(\lambda^{-3+o(1)} e^{\frac{2\pi n}{\lambda}}\right) = O\left(n^{-\frac{3\epsilon}{1+\epsilon}+o(1)}\right)$
$= o(1)$

Thus we have completed the proof.

yours
S. Ramanujan

(1)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\cdots\right)}$$

$$C = 1+4\left(\frac{1^7}{e^{1\pi}}+\frac{2^7}{e^{2\pi}}+\frac{3^7}{e^{3\pi}}+\cdots\right)$$

If n is a positive integer, then

(i) $\frac{1}{2}C\,p_n = \frac{C_1(n)}{1^4}e^{2n\pi} + \frac{C_2(n)}{2^4}e^{n\pi} + \frac{C_3(n)}{3^4}e^{\frac{2n\pi}{3}} + \cdots$

(ii) $\frac{1}{2}C\,u_n = \frac{C_1(n)}{1^3}e^{2n\pi} + \frac{C_2(n)}{2^3}e^{n\pi} + \frac{C_3(n)}{3^3}e^{\frac{2n\pi}{3}} + \cdots$

(iii) $\frac{1}{2}C\,v_n = \frac{C_1(n)}{1^2}e^{2n\pi} + \frac{C_2(n)}{2^2}e^{n\pi} + \frac{C_3(n)}{3^2}e^{\frac{2n\pi}{3}} + \cdots$

where $C_1(n)=1$, $C_2(n)=(-1)^n$, $C_3(n)=2\cos\left(\frac{2n\pi}{3}+\gamma(n-2)\right)$, ...

and

$$\sum u_n x^n = \frac{\pi}{3}\cdot\frac{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\cdots\right)}{1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\cdots\right)}$$

$$\sum v_n x^n = \frac{\pi^2}{9}\cdot\frac{\left\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\cdots\right)\right\}^2}{1-504\left(\frac{x}{1-x}+\frac{2^5x^2}{1-x^2}+\frac{3^5x^3}{1-x^3}+\cdots\right)}$$

If n is a negative integer, then

(i) $0 = \frac{C_1(n)}{1^4}e^{2n\pi} + \frac{C_2(n)}{2^4}e^{n\pi} + \frac{C_3(n)}{3^4}e^{\frac{2n\pi}{3}} + \cdots$

(ii) $0 = \frac{C_1(n)}{1^3}e^{2n\pi} + \frac{C_2(n)}{2^3}e^{n\pi} + \frac{C_3(n)}{3^3}e^{\frac{2n\pi}{3}} + \cdots$

(iii) $0 = \frac{C_1(n)}{1^2}e^{2n\pi} + \frac{C_2(n)}{2^2}e^{n\pi} + \frac{C_3(n)}{3^2}e^{\frac{2n\pi}{3}} + \cdots$

All these can be proved using the formula:

If $ad-bc=1$, $q = e^{i\pi\cdot\frac{c+id}{a+ib}}$, then

(I) $$\frac{\pi}{3}\left\{1-24\left(\frac{q^2}{1-q^2}+\frac{2q^4}{1-q^4}+\frac{3q^6}{1-q^6}+\cdots\right)\right\} = a^2+b^2.$$

There is only one more formula like (I) which is given at the end of the next page. Besides (II) and (III) there is no more possibility

$$\sum p_n x^n = \frac{1}{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \cdots\right)}$$

$$C = 1 + 504\left(\frac{1^5}{e^{n\pi\sqrt{3}}+1} - \frac{2^5}{e^{2n\pi\sqrt{3}}+1} + \cdots\right)$$

If n is 0 or a positive integer then

(i) $\frac{1}{3} C\, p_n = (-1)^n \left\{ \frac{c_1(n)}{1^3} e^{n\pi\sqrt{3}} + \frac{c_2(n)}{2^3} e^{\cdots} + \frac{c_7(n)}{7^3} e^{\cdots} + \cdots \right\}$

(ii) $\frac{1}{3} C\, u_n = (-1)^n \left\{ \frac{c_1(n)}{1^3} e^{n\pi\sqrt{3}} + \cdots + \frac{c_2(n)}{2^3} e^{n\pi\sqrt{3}} + \cdots \right\}$

where $c_1(n) = 1$, $c_2(n) = -1$, ... and

$$\sum u_n x^n = \frac{\pi}{2\sqrt{3}} \cdot \frac{1 - 24\left(\frac{x}{1-x} + \frac{2^3 x^2}{1-x^2} + \cdots\right)}{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \cdots\right)}$$

If n is a negative integer then

(i) $0 = \frac{c_1(n)}{1^3} e^{n\pi\sqrt{3}} + \frac{c_2(n)}{2^3} e^{\cdots} + \frac{c_7(n)}{7^3} e^{\cdots} + \cdots$

(ii) $0 = \frac{c_1(n)}{1^3} e^{n\pi\sqrt{3}} + \frac{c_2(n)}{2^3} e^{\cdots} + \frac{c_7(n)}{7^3} e^{\cdots} + \cdots$

These are proved using the formula:

If $ad - bc = 1$, $q = e^{i\pi \cdot \frac{c+\omega d}{a+\omega b}}$, $\omega = \frac{-1+i\sqrt{3}}{2}$,

(II) then $\frac{\pi}{2\sqrt{3}} \left\{ 1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right) \right\}$

$= a^2 - ab + b^2$

$$\sum_0^\infty b_n x^n = \frac{1}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{(1-x)^2} + \cdots\right.}$$

$b_0 =$ 1

$b_1 =$ 504

$b_2 =$ 270648

$b_3 =$ 144,912096

$b_4 =$ 77599,626552

$b_5 =$ 41,553943,041744

$b_6 =$ 22251,789971,649504

$b_7 =$ 11,915647,845248,387520

$b_8 =$ 6380,729991,419236,488504

$b_9 =$ 3,416827,666538,895485,479576

$b_{10} =$ 1829,682703,808504,464920,468048

$b_{11} =$ 979779,820147,442370,107345,764512

$b_{12} =$ 524,663917,940510,191509,934144,603104

524,663917,940510,190113,197271,938395·329

+ 1390,736872,662028·140

+ 2680·418

+ ·130

− ·014

− ·003

524,663917,940510,191509,934144,603104·000

(1)

Matlock House
Monday

Dear Mr Hardy,

Will my letters be forwarded to you during the [illegible] if I write to you to the Vicarity [illegible]

I considered another problem connected with [illegible] Suppose that

$$\sum p_n x^n = \frac{1}{1 - 50\,4\left(\frac{x^n}{1-x} + \frac{x^2 x^2}{1-x^2} + \cdots\right)}$$

and that $p_m = T_1^{(m)} + T_2^{(m)} + T_5^{(m)} + T_{10}^{(m)} + \cdots$

where the T's are the terms in the asymptotic expansion. Now $T_1^{(m)} + T_{[illegible]}$ is an integer for integral values of n. The question arises: if we separate the [illegible] even suffixes and [illegible] the two separate series

$$T_1^{(n)} + T_5^{(n)} + T_{13}^{(n)} + \cdots$$

and

$$T_2^{(n)} + T_{10}^{(n)} + T_{26}^{(n)} + \cdots$$

what is the nature of the values of each of these series? I did not succeed in solving this problem but at any rate I have arrived at the following result which [illegible]s from the identities I wrote to you in my [illegible] letter.

(2)

I have already written to you that—

if $\sum a_n x^n = \frac{1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \cdots\right)}{1 - 504\left(\frac{x^5}{1-x^5} + \frac{2x^{10}}{1-x^{10}} + \cdots\right)}$

then $a_n = \frac{3}{\pi}\left(T_1(n) + 2T_9(n) + 5T_{25}(n) + 10T_{10}(n) + \cdots\right)$

Suppose now that

$$\sum h_n x^n = \frac{1}{\left\{1 + 24\left(\frac{x}{1-x} + \frac{3x^3}{1-x^3} + \cdots\right)\right\}^2}$$

then $h_n = 18\left\{\left(n + \frac{3}{\pi}\right)T_2(n) + \left(n + \frac{15}{\pi}\right)T_{10}(n) + \left(n + \frac{39}{\pi}\right)T_{26}(n) + \cdots\right\}$,

It follows that

$$18\left\{\left(n + \frac{3}{2\pi}\right)T_1(n) + \left(n + \frac{15}{2\pi}\right)T_5(n) + \left(n + \frac{39}{2\pi}\right)T_{26}(n) + \cdots\right\}$$

is also an integer as we already know that

$T_1(n) + T_2(n) + \cdots$ is an integer

and $\frac{3}{\pi}\left\{T_1(n) + 2T_4(n) + 5T_{25}(n) + \cdots\right\}$ is an integer and consequently

$$2\left\{\left(n + \frac{3}{2\pi}\right)T_1(n) + \left(n + \frac{3}{\pi}\right)T_2(n) + \left(n + \frac{15}{2\pi}\right)T_5(n) + \left(n + \frac{15}{\pi}\right)T_{10}(n) + \cdots\right\}$$

is an integer. Hence the result.

(2)

I am now considering another problem I told you — viz the expression of the various powers of $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{1}{8})$ as products in primes of the powers of $\Gamma(\frac{1}{8})$ i.e the powers of π. For instance we know that

$$\pi^2\left(1-\tfrac{1}{2^2}\right)\left(1-\tfrac{1}{3^2}\right)\left(1-\tfrac{1}{5^2}\right)\left(1-\tfrac{1}{7^2}\right)\cdots = 6$$

$$\pi^4\left(1-\tfrac{1}{2^4}\right)\left(1-\tfrac{1}{3^4}\right)\left(1-\tfrac{1}{5^4}\right)\left(1-\tfrac{1}{7^4}\right)\cdots = 90$$

$$\pi^6\left(1-\tfrac{1}{2^6}\right)\left(1-\tfrac{1}{3^6}\right)\left(1-\tfrac{1}{5^6}\right)\cdots = 945$$

$$\pi^8\left(1-\tfrac{1}{2^8}\right)\left(1-\tfrac{1}{3^8}\right)\left(1-\tfrac{1}{5^8}\right)\cdots = 9450$$

$$\pi^{10}\left(1-\tfrac{1}{2^{10}}\right)\left(1-\tfrac{1}{3^{10}}\right)\left(1-\tfrac{1}{5^{10}}\right)\cdots = 93555$$

and so on, the right-hand side being a rational number. I find that

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{1}{4}}}\right\}^4\left(1-\tfrac{1}{3^2}\right)\left\{\left(1-\tfrac{1}{(1+2i)^2}\right)\left(1-\tfrac{1}{(1-2i)^2}\right)\right\}$$
$$\times\left(1-\tfrac{1}{7^2}\right)\left(1-\tfrac{1}{11^2}\right)\left\{\left(1-\tfrac{1}{(3+2i)^2}\right)\left(1-\tfrac{1}{(3-2i)^2}\right)\right\}$$
$$\times\left\{\left(1-\tfrac{1}{(1+4i)^2}\right)\left(1-\tfrac{1}{(1-4i)^2}\right)\right\}\left(1-\tfrac{1}{19^2}\right)\cdots\cdots$$
$$= 4$$

Here we take the odd primes 3, 5, 7, 11, 13, 17, 19 .... and retain the primes of the form $4k-1$ viz 3, 7, 11, 19 — and instead of the primes of the form $4k+1$ viz 5, 13, 17, ... we write the corresponding factors $1+2i$ and $1-2i$; $3+2i$ and $3-2i$; $1+4i$ and $1-4i$ and so on

(4)

It should be remembered that the odd number must be the real parts and the even number the imaginary one in the complex factors. For instance $29 = 2^2 + 5^2$ and $37 = 1^2 + 6^2$. The conjugate factors corresponding to 29 and 37 must be written as $5+2i$ and $5-2i$; $1+6i$ and $1-6i$ and not as $2+5i$ and $2-5i$, With the same convention I find that

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{1}{4}}}\right\}^8 (1-\tfrac{1}{3^4})\left\{(1-\tfrac{1}{(1+2i)^4})(1-\tfrac{1}{(1-2i)^4})\right\}$$
$$\times(1-\tfrac{1}{5^4})(1-\tfrac{1}{11^4})\cdots = 12;$$

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{1}{4}}}\right\}^{12} (1-\tfrac{1}{3^8})\left\{(1-\tfrac{1}{(1+2i)^6})(1-\tfrac{1}{(1-2i)^6})\right\}$$
$$\times(1-\tfrac{1}{7^6})(1-\tfrac{1}{11^6})\cdots = 40;$$

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{1}{4}}}\right\}^{16} (1-\tfrac{1}{3^8})\left\{(1-\tfrac{1}{(1+2i)^8})(1-\tfrac{1}{(1-2i)^8})\right\}$$
$$\times(1-\tfrac{1}{7^8})(1-\tfrac{1}{11^8})\cdots = 140;$$

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{1}{4}}}\right\}^{20} (1-\tfrac{1}{3^{10}})\left\{(1-\tfrac{1}{(1+2i)^{10}})(1-\tfrac{1}{(1-2i)^{10}})\right\}$$
$$\times(1-\tfrac{1}{7^{10}})(1-\tfrac{1}{11^{10}})\cdots = 480;$$

$$\left\{\frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{3}{4}}}\right\}^{24}\left(1-\frac{1}{3^{12}}\right)\left(1-\frac{1}{(1+2i)^{12}}\right)\left(1-\frac{1}{(1-2i)^{12}}\right)$$
$$\times\left(1-\frac{1}{7^{12}}\right)\left(1-\frac{1}{17^{12}}\right)\cdots = 1650;$$

and so on. I am confident that I will certainly get similar results for the powers of $\Gamma(\frac{1}{8})$ and $\Gamma(\frac{1}{8})$ is substantially $\Gamma(\frac{1}{4})$ since

$$\Gamma(\tfrac{1}{8})\sqrt{(\tfrac{\pi}{3})}\sqrt[8]{2} = \{\Gamma(\tfrac{1}{4})\}^2.$$

But I am not sure whether I'll succeed in dealing with the powers of $\Gamma(\frac{1}{8})$ or not. Could you tell me if any such results were known before?

yours ever

S. Ramanujan.

If s is a multiple of 4. Add the $\zeta(s)$ (1)

$$\zeta(s) = \frac{1}{1-(1+i)^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-(1+2i)^{-s}} \cdot \frac{1}{1-(3+2i)^{-s}}$$

$$= \frac{1}{2}\zeta(s) + \frac{(2\pi)^s}{2\Gamma(s)}\left(\frac{1^{s-1}}{e^{2\pi i}-1} + \frac{2^{s-1}}{e^{4\pi i}-1} + \cdots\right)$$

If $\zeta(s) = \lambda_8 \frac{\{\Gamma(\frac{1}{4})\}^{2s}}{5^{\frac{1}{2}s}(8\pi)^{\frac{1}{2}s}}$. Then $\lambda_4 = +2$, $\lambda_8 = \frac{1}{84}$, 77

$$\lambda_{12} = \frac{1}{8.11.13}, \quad \lambda_{16} = \frac{1}{17.18.26}, \quad \lambda_{20} = \frac{1}{13.17.18.19}$$

$$\lambda_{24} = \frac{5}{13.17.23.26.27}, \quad \lambda_{28} = \frac{1}{13.17.27.29.39}$$

$$\lambda_{32} = \frac{223}{17.26.27.29.31.34.39}$$

$$\pi x\left\{\frac{1}{2}x + \frac{1}{2}\cot \pi x + \frac{\sin 2\pi x}{e^{2\pi i}-1} + \frac{\sin 4\pi x}{e^{4\pi i}-1} + \cdots\right\}$$

$$= \frac{1}{2} - x^4 z(4) - x^8 z(8) - x^{12} z(12) - \cdots$$

$$12\left\{3x^4 z(4) + 7x^8 z(8) + 11 x^{12} z(12) + \cdots\right\}^2$$

$$= 1.7.9\, x^8 z(8) + 3.11.13\, x^{12} z(12) + 5.15.17\, x^{16} z(16) + \cdots$$

$$\lambda_{36} = \frac{61}{3.13.17.26.29.37.39.51}$$

$$\lambda_{40} = \frac{2381}{13.17.26.27.29.37.39.41.51}$$

$$\lambda_{44} = \frac{1805}{3.13.17.29.37.39.39.41.43.51}$$

$$\lambda_{48} = \frac{581126\,5}{17.21.29.34.37.39^4.41.47.51}$$

$$\lambda_{52} = \frac{8887}{3^5.7.13.17^3.29.37.41.53}$$

$$\lambda_{56} = \frac{3189629}{3^7.7.13^4.17^3.29^2.37.41.53}$$

$$\lambda_{60} = \frac{2.263096660\,33}{3^6.7.13^5.17^3.29^2.37.41.53.59.61}$$

$$a_{28} = \frac{113}{3636140625}$$

$$a_{26} = \frac{15\cdot 11}{12144000000}$$

$$a_{30} = \frac{2603099}{28794335\,2000000}$$

A Sheet from the 'LOST NOTEBOOK'

$(1-x^4)(1+\frac{x^4}{2^4})(1-\frac{x^4}{3^4})\cdots$ (not primes)

$= e^{\frac{1}{2}\pi x^2}(\sin\pi x - e^{-2\pi}\sin 3\pi x + e^{-6\pi}\sin 5\pi x - \cdots)$

$= \dfrac{}{\pi x(1-3e^{-2\pi}+5e^{-6\pi}\cdots)}$

If s is an even integer then

$z_1(s) = \dfrac{1}{1-3^{-s}}\cdot\dfrac{1}{1-(1+2i)^{-s}}\cdot\dfrac{1}{1-(1-2i)^{-s}}\cdots$

(odd numbers first).

$\frac{\pi}{2\sqrt{2}}\left\{\frac{\sin\frac{\pi x}{\sqrt{2}}}{\cosh\frac{1}{2}\pi} - \frac{\sin\frac{3\pi x}{\sqrt{2}}}{\cosh\frac{3}{2}\pi} + \cdots\right\}$

$= x\,z_1(2) - x^5 z_1(6) + x^9 z_1(10) - \cdots$

$\frac{\pi x}{8} - \frac{\pi}{2\sqrt{2}}\left\{\frac{\sin\pi x\sqrt{2}}{\sinh\pi} - \frac{\sin 2\pi x\sqrt{2}}{\sinh 2\pi} + \cdots\right\}$

$= x^3 z_1(4) - x^7 z_1(8) + x^{11} z_1(12) - \cdots$

$(1+\frac{x^4}{3^4})(1+\frac{x^4}{(1+2i)^4})(1+\frac{x^4}{(1-2i)^4})\cdots$

$= e^{\frac{\pi x^2}{4}}\dfrac{(1+2e^{-\pi}\cos\pi x\sqrt{2}+2e^{-4\pi}\cos 2\pi x\sqrt{2}+\cdots)}{(1+2e^{-\pi}+2e^{-4\pi}+\cdots)}$

$(1+\frac{ix^2}{3^2})(1+\frac{ix^2}{(1+2i)^2})(1+\frac{ix^2}{(1-2i)^2})\cdots$

$= e^{\frac{\pi x^2}{4}}\dfrac{(1+2ie^{-\frac{1}{2}\pi}\cos\frac{\pi x}{\sqrt{2}}+2e^{-2\pi}\cos\pi x\sqrt{2}+\cdots)}{(1+2ie^{-\frac{1}{2}\pi}+2e^{-2\pi}+\cdots)}$

$(1-\frac{ix^2}{3^2})(1-\frac{ix^2}{(1+2i)^2})(1-\frac{ix^2}{(1-2i)^2})\cdots$

$= e^{\frac{\pi x^2}{8}}\dfrac{1+2ie^{-\frac{1}{2}\pi}\cos\frac{\pi x}{\sqrt{2}}+2e^{-2\pi}\cos\pi x\sqrt{2}+\cdots}{1+2ie^{-\frac{1}{2}\pi}+2e^{-2\pi}+\cdots}$

$4\{x^2 z_1(2) + x^6 z_1(6) + x^{10} z_1(10) + \cdots\}^2$

$= 3x^4 z_1(4) + 7x^8 z_1(8) + 11x^{12} z_1(12) + \cdots$

$z_1(s) = a_s\dfrac{[\Gamma(\frac{1}{4})]^{2s}}{(16\pi)^s}$

$a_2 = \frac{1}{4}$, $a_4 = \frac{1}{12}$, $a_6 = \frac{1}{40}$, $a_8 = \frac{1}{140}$,

$a_{10} = \frac{1}{480}$, $a_{12} = \frac{1}{1650}$, $a_{14} = \frac{11}{67400}$,

$a_{16} = \frac{1}{19500}$, $a_{18} = \frac{211}{14144000}$,

$a_{20} = \frac{41}{9447750}$, $a_{22} = \frac{7}{19861250}$

$a_{22} = \frac{1607}{1372960000}$

28th June 1918

Dear Mr Hardy,

I sent you the proofs. In 1.11 you say that the partition method gives exact values only in the case of 4, 6 and 8 squares. What about 3, 5, 7 squares? Are they not exact?

I am sending you the analogous [illegible] 9. Please mention these in the paper [illegible]. After all we have got only two exact examples to offer, viz 2 and 4. So please don't omit the remarks.

You can write many things at the foot-notes instead of the body of the paper so that they will occupy very much less space. I made a slight mistake in the proof of the least no. of terms; instead of writing $2\cos\frac{k}{p}k n\pi + \theta(-1)^n$, $2\cos\left(\frac{2k(n-1)\pi}{p} + 2\theta(-1)^{\theta}\right)$, $2\cos\frac{2k(n-2)\pi}{p}$, ... where k is a positive integer less than p. I omitted k in my letter. I don't think the arguments are affected. Are they?

Recently I received my scholarship from the Madras University. I am sending you with this letter a cheque for £100 for the present.

Ever yours

S. Ramanujan

(1)

The result in the case of ϑ_3 is

$$C = 1 + 480\left(\frac{1^7}{e^{i\pi\tau} - 1} + \frac{2^7}{e^{2i\pi\tau} - 1} + \cdots\right),$$

$$ad - bc = 1, \quad \frac{a}{c} = e^{i\pi \cdot \frac{a+id}{c+ib}},$$

$$1 - 504\left(\frac{1^5 q^1}{1-q^1} + \cdots\right) = \frac{1}{1-q^2 e^{i\pi}} + \frac{1}{2^2}\cdot\frac{1}{1+q^2 e^{i\pi}} + \cdots$$

$$= \sum \frac{1}{(a+ib)^4} \cdot \frac{1}{1 - \frac{q^2}{e^{2i\pi\frac{a}{c}}}}.$$

The analogous result in the case ϑ_2 is

$$C = 1 + 504\left(\frac{1^5}{e^{i\pi\sqrt{3}+1}} - \frac{2^5}{e^{2i\pi\sqrt{3}}+1} + \frac{3^5}{e^{3i\pi\sqrt{3}}+1} - \cdots\right),$$

$$ad - bc = 1, \quad \omega = \frac{-1+i\sqrt{3}}{2}, \quad \frac{a}{c} = e^{i\pi \cdot \frac{a+\omega d}{c+\omega b}}$$

(1) $\frac{1}{3}C$

$$1 + 240\left(\frac{1^3 q^1}{1-q^1} + \frac{2^3 q^2}{1-q^2} + \cdots\right) = 1 + q^2 e^{i\pi\sqrt{3}} - \frac{1}{3^3}\cdot\frac{1}{1+q^2 e^{i\pi\sqrt{3}}} + \cdots$$

$$= \sum \frac{1}{(a+\omega b)^6} \cdot \frac{1}{1 - \frac{q^2}{e^{2i\pi\frac{a}{c}}}}.$$

$$C = \frac{32}{11}\left\{1 - 504\left(\frac{1^5}{e^{i\pi\sqrt{3}}-1} + \frac{2^5}{e^{2i\pi\sqrt{3}}-1} + \frac{3^5}{e^{3i\pi\sqrt{3}}-1} + \cdots\right)\right\}^{12}$$

$$= (2\sqrt{3})^3 e^{-\frac{1}{2}\pi\sqrt{3}}\left\{(1+e^{-\pi\sqrt{3}})(1-e^{-2\pi\sqrt{3}})(1+e^{-3\pi\sqrt{3}})\cdots\right\}^{12}$$

$$= 2(4\sqrt{3})^3 e^{-\pi\sqrt{3}}\left\{(1-e^{-2\pi\sqrt{3}})(1-e^{-4\pi\sqrt{3}})\cdots\right\}^{12}.$$

$$= \left(\frac{\pi}{3}\right)^{\frac{3}{2}} \cdot \frac{2^3}{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^9} = \left(\frac{\pi}{3}\right)^6 \cdot \frac{2^9}{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^{18}}.$$

$$\sum p_n x^n = \frac{1}{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \cdots\right)}$$

(2) $\frac{1}{3}C\,p_n = (-1)^n\left\{c_1 e^{n\pi\sqrt{3}} + c_2 e^{\frac{n\pi\sqrt{3}}{3}} + c_7 e^{\frac{n\pi\sqrt{3}}{7}} + \frac{c_{13}}{13^3} e^{\frac{n\pi\sqrt{3}}{13}} + \cdots\right\}$

$= (-1)^n \sum \frac{c_\lambda}{\lambda^3} e^{\frac{n\pi\sqrt{3}}{\lambda}}$

when 1, 3, 7, 13 are the numbers of the form $a^2 + ab + b^2$, a and b being prime to each other. That is to say, 1, 3, 7, 13, ... are of the form λ.

$$3^{a_3} \cdot 7^{a_7} \cdot 13^{a_{13}} \cdot 19^{a_{19}} \cdots$$

where $a_3 = 0$ or 1 and $a_7, a_{13}, \ldots$ are zeros or positive integers, $7, 13, 19, \ldots$ being primes of the form $6k+1$. The first 20 values of n are

$1, 3, 7, 13, 19, 21, 31, 37, 39, 43,$
$49, 57, 61, 67, 73, 79, 91, 93, 97, 103.$

$$c_1 = 1, \quad c_3 = -1, \quad c_7 = 2\cos\left(\frac{2n\pi}{7} + 6\tan^{-1} 3\sqrt{3}\right)$$

$$c_{13} = 2\cos\left(\frac{6n\pi}{13} + 6\tan^{-1} 2\sqrt{3}\right),$$

$$c_{19} = \pm 2\cos\left(\frac{4n\pi}{19} - 6\tan^{-1} 5\sqrt{3}\right)$$

$$c_{21} = -2\cos\left(\frac{4n\pi}{7} + 6\tan^{-1} 3\sqrt{3}\right)$$

If n is a negative integer the right-hand side of (2) is zero. If $|x| \geq 1$, the right-hand side of (1) is zero. The [illegible] are exactly similar to those for g_3.

Mat-Lock Hous[illegible]

[illegible]

Dear Mr Hardy, I have been unwell since you saw me except for 2 or 3 days. It is only to day I feel so much better.

I conclude about $\frac{9}{2}$: It presents some interesting features. In the first place it is easy to prove the following identities

Suppose that

$$\phi(x) = 1 + 240\left(\frac{x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \ldots\right).$$

then

(1) $\phi(\sqrt{x}) + \phi(-\sqrt{x}) = 2\phi(x)$

(2) $\phi(\sqrt{x})\,\phi(-\sqrt{x})\,\phi(x) = 1 - 504\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \ldots\right)$

(3) $5\{\phi(x)\}^2 = 1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \ldots\right)$
$+ 4\left\{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \frac{2^3 x^4}{1-x^4} + \ldots\right)\right\}$

(4) $7\{\phi(x)\}^3 = -\left\{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \ldots\right)\right\}$
$+ 8\left\{1 - \ldots 4\left(\frac{1^5 x^2}{1-x^2} + \frac{2^5 x^4}{1-x^4} + \ldots\right)\right\}$

(5) $\frac{2}{3}\left\{\frac{1}{\phi\sqrt{x}} + \frac{1}{\phi(-\sqrt{x})}\right\} - \frac{1}{3\phi(x)}$
$= \dfrac{1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \ldots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \ldots\right)}$

(6) $\frac{11}{24}\left\{\frac{1}{\phi(\sqrt{x})} + \frac{1}{\phi(-\sqrt{x})}\right\} + \frac{1}{12\phi(x)}$
$= \dfrac{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \frac{2^3 x^4}{1-x^4} + \ldots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \ldots\right)}$

(2)

Also we have

(7) $15(1-2x+2x^4-2x^9+\dots)^8$

$= -\{1+240(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\dots)\}$

$+16\{1+240(\frac{1^3x^2}{1-x^2}+\frac{2^3x^4}{1-x^4}+\dots)\}$

(8) $240x(1+x+x^3+x^6+\dots)^8$

$= 1+240(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\dots)$

$-\{1+240(\frac{1^3x^2}{1-x^2}+\frac{2^3x^4}{1-x^4}+\dots)\}$.

Suppose now that

$$\sum a_n x^n = \frac{1+240(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\dots)}{1-504(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\dots)}$$

then

$$a_n = \frac{2}{C}\left\{e^{2n\pi}-\frac{(-1)^n}{2^2}e^{n\pi}+\frac{2\cos(\frac{2n\pi}{5}+4\tan^{-1}2)}{5^2}\left(e^{\frac{2n\pi}{5}}+\dots\right)\right\}$$

$$= \frac{2}{C}\sum \frac{1}{(a+ic)^4}\cdot e^{-2in\pi\frac{a+ic}{d}}$$

where $ad-bc=1$ and $C=1+240(\frac{1^3}{e^{2\pi}-1}+\frac{2^3}{e^{4\pi}-1}+\dots)$.

Say then $a_n = T_1+T_2+T_5+T_{10}+T_{13}+\dots$ where $T_1, T_2, T_5, \dots$ are the terms in the elliptic series. The following interesting identities follow from the identities (1)–(8).

If $\sum b_n x^n = \frac{1}{1+240(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\dots)}$

then $b_n = -3(T_2+T_{10}+T_{26}+\dots)$

where 2, 10, 26 &c are the even suffixes occurring in $a_n = T_1+T_5+T_{10}+\dots$

(8)

If $\sum g_n x^n = \dfrac{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \frac{2^3 x^4}{1-x^4} + \cdots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \cdots\right)}$

then $g_n = \frac{11}{18}(T_1 + T_5 + T_{13} + \cdots)$

$- \frac{1}{4}(T_2 + T_{10} + T_{26} + \cdots)$

If $\sum h_n x^n = \dfrac{(1 - 2x + 2x^4 - \cdots)^8}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \cdots\right)}$

then $h_n = \frac{3}{3}(T_1 + T_5 + T_{13} + \cdots) - \frac{1}{3}(T_2 + T_{10} + \cdots)$

If $\sum l_n x^n = \dfrac{x(1 + x + x^3 + \cdots)^8}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \cdots\right)}$

then $l_n = \frac{1}{768}(T_1 + T_5 + T_{13} + \cdots) + \frac{1}{192}(T_2 + T_{10} + \cdots)$

If n is a negative integer then

$T_1 + T_2 + T_5 + T_{10} + T_{13} + \cdots = 0$;

$T_1 + T_5 + T_{13} + \cdots = 0$; $T_2 + T_{10} + T_{26} + \cdots = 0$.

Of all these the coefficients in

$\dfrac{1}{1 + 24\left(\frac{x}{1-x} + \frac{3x^3}{1-x^3} + \cdots\right)} = \sum l_n x^n$

are the most interesting and the same numerical approximations must be more strikingly remarkable than in $1/g_2$ & $1/g_3$ problems since $l_n = -\frac{1}{3}(T_2 + T_{10} + T_{26} + \cdots)$ and T_{10}, T_{26} are very small.

Ever yours

S. Ramanujan

$$\sum \lambda_n x^n = \frac{1 + 240\left(\frac{1^3 x}{1-x} + \cdots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)} -$$

$$\lambda_n = 1 + T_2 + T_3 + T_{10} + T_{13} +$$

$$\frac{1}{1 + 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \cdots\right)} \longrightarrow -3\,(T_2 + T_{10} + T_{26} + \cdots)$$

$$\frac{1 + 240\left(\frac{1^3 x}{1-x} + \cdots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)} = \frac{11}{12}(T_1 + T_3 + T_{13} + \cdots) - \frac{1}{12}(T_2 + T_{10} + T_{26} + \cdots)$$

$$\frac{(1 - 2x + 2x^4 - 2x^9 + \cdots)^8}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)} = \frac{2}{3}(T_1 + T_3 + \cdots) - \frac{1}{3}(T_2 + T_{10})$$

$$\frac{x(1 + x + x^3 + \cdots)^{24}}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)} = \frac{1}{780}(T_1 + T_3 + \cdots) + \frac{1}{192}(T_2 + T_{10})$$

$$\phi(x) = \frac{1}{1 + 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{2x^3}{1-x^3} + \cdots\right)}$$

$$\frac{2}{3}\{\phi(\sqrt{x}) + \phi(-\sqrt{x})\} - \frac{1}{3}\phi(x) = \frac{1 + 240\left(\frac{1^3 x}{1-x} + \cdots\right)}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)}$$

$$\frac{11}{24}\{\phi(\sqrt{x}) + \phi(-\sqrt{x})\} + \frac{1}{12}\phi(x) = \frac{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \cdots\right)}{1 - 504\left(\frac{1^5 x^2}{1-x^2} + \cdots\right)}$$

$$\frac{6}{\{\phi(x)\}^2} = \left\{1 + 240\left(\frac{1^3 x}{1-x} + \cdots\right)\right\} + 4\left\{1 + 240\left(\frac{1^3 x^2}{1-x^2} + \cdots\right)\right\}$$

$$\frac{7}{\{\phi(x)\}^3} = -\left\{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)\right\} + 8\left\{1 - 504\left(\frac{1^5 x^2}{1-x^2} + \cdots\right)\right\}$$

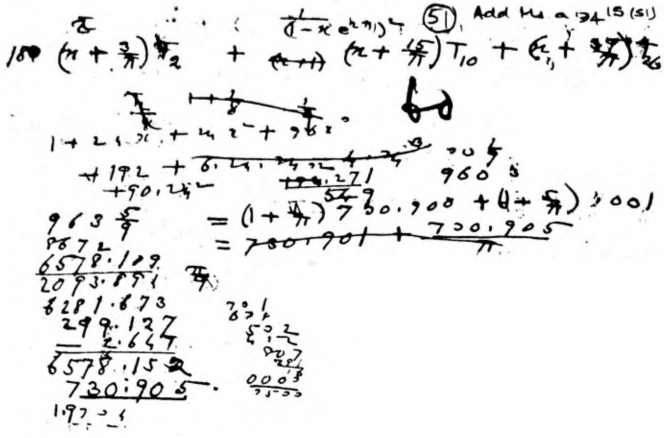
$$\phi(x)\phi(-x)\phi(x^2) = \frac{1}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \cdots\right)}$$

$$\sum \tau_n x^n = \frac{1}{1 - 504\left(\frac{1^5 x}{1-x} + \cdots\right)}$$

$$\tau_n = T_1(-1) + T_2(-1) + T_3(-1) + T_{10}(-1)$$

$$\sum \nu_n x^n = \frac{1}{\left\{1 + 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \cdots\right)\right\}^2}$$

where $\nu_n = 18\left\{\left(n + \frac{3}{\pi}\right)T_2(-n) + \left(n + \frac{15}{\pi}\right)T_{10}(-n) + \cdots\right\}$



Trinity Home
11 Trinity Square
Monday

Dear Mr Hardy,

Please tell Mr Littlewood and Major Mac Mahon that I thanked them very much. Had it not been for your pains and their encouragement I would be neither the fellow of the one nor the other. Have the tutors and the Master of Trinity more voice in the election than the other fellows? I heard that in some colleges that there are two kinds of fellowships: one lasting for two or three years and the other for five or six years; if that is so in Trinity, do you mean the first or the second kind?

I have considered more or less exhaustively about the congruency of $p(n)$ and in general that of b_n where $\sum p(n) x^n = 1/\{(1-x)(1-x^2)(1-x^3)\ldots\}$. I have four different methods. Each method has its own advantages. In the case of $p(n)$ the results are roughly these:

I method — Very special and very simple. Gives only $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$

This you are publishing now.

II. D. function method. This is general theoretically but becomes practically unworkable due to tediousness of calculation with the exception of few cases. I have simplified the proof very much and made it quite elementary

is established quite easily

as the first method. From this the divisibility by 5, 5^2, 5^3 and 7, 7^2, 7^3 I have not considered the case of 11 due to its advanced ness. This method gives many more results like (17) and (18) in the proof sheet.

III Dirichlet's Series method. This is very general and practically workable. The conjectured products of $\mathfrak{D}$ Series for corresponding to $\{(1-x)(1-x^2)(1-x^3)\}^m$ when $m = 2, 4, 6, 8, 12, 24$ were proved by Mr Mordell. I have now conjectured analogous results for some more values of m viz. $10, 14, 16, 18, 20, 22, 28, 30, 32, 36, 48$. All these [illegible] be proved by Mr Mordell. If we assume these results this method gives information about the divisibility by of $p(n)$

5, 5^2, 7, 7^2, 11, 13, 17, 19, 23, 29, 31 and 37.

IV g_2, g_3 method. This is very general. This except the case of 5^3, 7^3 gives all the previous results and the divisibility by 11^2, as well. Thus the divisibility by $5^a 7^b 11^c$ when $a = 0, 1, 2, 3$; $b = 0, 1, 2, 3$; $c = 0, 1, 2$ amounting to $4 \times 4 \times 3 - 1$ or 47 cases of the conjectured theorem are proved.

Ever yours

S. Ramanujan

S is a number which can be expressed as the sum of 2 squares which are prime to each other. That is to say S is of the form

$$2^{\lambda}.5^{a}.13^{b}.17^{c} \qquad \lambda = 0 \text{ or } 1.$$

The values of S not exceeding 200 are

1, 2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, 50, 53, 58, 61, 65, 73, 74, 82, 85, 89, 97, 101, 106, 109, 113, 122, 125, 130, 137, 145, 146, 149, 157, 169, 170, 173, 178, 181, 185, 193, 194, 197.

---

u and v are any 2 positive numbers which are prime to each other. ρ and γ are any two integers such that

$$\rho u - \gamma v = 1.$$

$k_{u,v}$ is an integer such that u^2+v^2 is a divisor of

$$\rho v + \gamma u + k_{u,v}.$$

$$\frac{\theta}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \cdots\right)}$$

$$= \frac{1}{1 - x e^{2\pi}} + \sum_{u,v} \frac{1}{(u+iv)^8} \frac{1}{1 - x e^{\frac{2\pi(1+i k_{u,v})}{u^2+v^2}}}$$

$$= \frac{1}{1 - x e^{2\pi}} + \sum_{u,v} \frac{1}{(u+iv)^8} \frac{1}{1 - x e^{\frac{2\pi(1+i k_{u,v})}{u^2+v^2}}},$$

where $\theta = \frac{1}{2}\left\{1 + 480\left(\frac{1^7}{e^{1\pi}-1} + \frac{2^7}{e^{2\pi}-1} + \frac{3^7}{e^{3\pi}-1} + \cdots\right)\right\}$

$$\sum_{0}^{\infty} \phi(n) x^n = \frac{\theta}{1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \cdots\right)}$$

$$\phi(n) \frac{1}{14} e^{2n\pi} + \frac{c_2}{13} e^{n\pi} + \frac{c_5}{5^4} e^{\frac{2n\pi}{5}} + \frac{c_{10}}{10^4} e^{\frac{2n\pi}{5}} + \cdots$$

$c_1 = 1$, $c_2 = (-1)^n$, $c_5 = 2\cos\left(\frac{4n\pi}{5} + 8\tan^{-1}2\right)$, $= c_{10}$.

$c_{13} = 2\cos\left(\frac{10n\pi}{13} + 8\tan^{-1}\frac{3}{2}\right)$ $c_{17} = 2\cos\left(\frac{8n\pi}{17} + 8\tan^{-1}4\right)$

$$\pm c = e^{\cdot 0879\,1178\ldots} \qquad (1)$$

$$\log c = \frac{c_1}{1^4} e^{2\pi\pi} + \frac{c_2}{2^4} e^{\pi\pi} + \frac{c_5}{5^4} e^{\frac{2\pi\pi}{5}} + \frac{c_{10}}{10^4} e^{\frac{2\pi\pi}{10}} + \frac{c_{13}}{13^4} e^{\frac{2\pi\pi}{13}} + \cdots = \sum_{\lambda} \frac{c_\lambda}{\lambda^4} e^{\frac{2\pi\pi}{\lambda}},$$

where λ is the series of 2 squares that are prime to each other, that is to say λ is a number of the form

$$2^{a_2} 5^{a_5} 13^{a_{13}} 17^{a_{17}} 29^{a_{29}} \ldots$$

(2) where $a_2 = 0$ or 1 and $a_5, a_{13}, a_{17}, \ldots$ are zeroes or positive integers, 5, 13, 17 ... being primes of the form $4k+1$. The first 30 values of λ are

1	17	37	61	85	109
2	25	41	65	89	113
5	26	50	73	97	122
10	29	53	74	101	125
13	34	58	82	106	130.

(3) $$c = 1 + 480\left(\frac{1^7}{e^{1\pi}-1} + \frac{2^7}{e^{4\pi}-1} + \frac{3^7}{e^{9\pi}-1} + \cdots\right)$$

$$\left(\frac{16}{\pi}\right)^8 = \frac{256}{125}\left\{1 + 480\left(\frac{1^7}{e^{4\pi}-1} + \frac{2^7}{e^{8\pi}-1} + \frac{3^7}{e^{12\pi}-1} + \cdots\right)\right\}^{16}$$

$$= 1\cdot 4 \; e^{-\frac{4\pi}{3}} \left\{(1-e^{-2\pi})(1-e^{-4\pi})(1-e^{-6\pi})\ldots\right\}^{16}$$

$$= 96^2 \, e^{-\frac{2\pi}{3}} \left\{(1-e^{-4\pi})(1-e^{-8\pi})(1-e^{-12\pi})\ldots\right\}^{16}.$$

$$\left(\frac{3}{2}\right)^2 = \frac{9}{16}\left\{\frac{\pi^4}{\Gamma(\frac{1}{4})}\right\}^{16}$$

Gauss has given (in the volume in which he deals with lemniscate functions) the values of $e^{-\pi}, e^{-2\pi}, e^{-3\pi}, e^{-4\pi}, \ldots e^{\pi}, e^{2\pi}, \ldots$ to a large number of decimal places. The reader can easily find the value of c and $\log c$ to 50 decimal places from the above series and products for c. Gauss has himself calculated the value of a root of c to a good many decimal places (in the same volume).

$c_1 = 1$; $c_2 = (-1)^m$,

$c_5 = 2\cos\left(\frac{4m\pi}{5} + 8\tan^{-1}2\right)$,

$c_{10} = 2\cos\left(\frac{2m\pi}{5} - 8\tan^{-1}2\right)$

$c_{13} = 2\cos\left(\frac{10m\pi}{13} + 8\tan^{-1}5\right)$

$c_{17} = 2\cos\left(\frac{8m\pi}{17} + 8\tan^{-1}4\right)$

$c_{25} = 2\cos\left(\frac{14m\pi}{25} + 8\tan^{-1}7\right)$

$c_{26} = 2\cos\left(\frac{5m\pi}{13} + 8\tan^{-1}5\right)$

$c_{34} = 2\cos\left(\frac{12m\pi}{17} - 8\tan^{-1}4\right)$

$c_{29} = 2\cos\left(\frac{24m\pi}{29} + 8\tan^{-1}\frac{5}{2}\right)$

$c_{37} = 2\cos\left(\frac{12m\pi}{37} + 8\tan^{-1}6\right)$

$c_{41} = 2\cos\left(\frac{16m\pi}{41} + 8\tan^{-1}\frac{5}{4}\right)$

$c_{50} = 2\cos\left(\frac{7m\pi}{25} + 8\tan^{-1}7\right)$

c_λ is the sum of $2^{\delta-1}$ cosines, where δ is the number of odd prime divisors of λ: the sum T of $2^{\delta-1}$ cosines [illegible]

a product of 8 cosines. Not only the number of cosines is reduced very much in the product but the alternating figures in the sum are reduced to great small figures in the product. e.g.

$$C_{65} = C_{5.13} = 2\cos\left(\frac{2n\pi}{5} - 8\tan^{-1}3\right).\ 2\cos\left(\frac{2n\pi}{13} + 8\tan^{-1}\frac{1}{3}\right)$$

$$C_{85} = C_{5.17} = 2\cos\left(\frac{2n\pi}{5} - 8\tan^{-1}2\right).\ 2\cos\left(\frac{2n\pi}{17} + 8\tan^{-1}\frac{1}{4}\right)$$

$$C_{105} = C_{5.13.17} = 2\cos\left(\frac{2n\pi}{5} - 8\tan^{-1}2\right).\ 2\cos\left(\frac{6n\pi}{13} - 8\tan^{-1}\frac{5}{12}\right)$$

$$\cdot\ 2\cos\left(\frac{2n\pi}{17} + 8\tan^{-1}\frac{1}{4}\right)$$

the equation (1). The value of the right hand side has been completely determined for all positive integral values of n including zero. When n is a negative integer the series is absolutely convergent as before. A very interesting question arises: What is its value for negative integral values? We shall now prove that its value is zero for every negative integral value of n.

Now

$$\frac{\frac{1}{2}c}{1 - \frac{1}{2}c\left(\frac{1}{1-x} + \frac{1}{1-x^2} + \cdots\right)}$$

$$= \frac{1}{1 - xe^{i\pi}} + \frac{1}{2^2(1 + xe^{i\pi})} + \cdots$$

$$= \frac{1}{1 - xe^{i\pi}} + \frac{1}{(1+i)^8\left(1 - xe^{\frac{i\pi}{1+i}}\right)} + \frac{1}{(1+i)^8\left(1 - xe^{\frac{2i\pi}{1+i}}\right)}$$

$$+ \frac{1}{(1-i)^8\left(1 - xe^{\frac{2\pi}{1-i}}\right)} + \frac{1}{(1+i)^8\left(1 - xe^{\frac{3\pi}{1+i}}\right)}$$

504 I
270,648 II
144,912,096 III
77,579,626,552 IV
41,553,743,041,744 V
22,251,789,971,649,504 VI

$a_3 = 144,912,096.000$

$a_2 = 270,648.000$

$a_1 = 504.000$

$a_0 = 1.000$

If we consider the ϑ-functions in the transformed [illegible] Eulerian e.g.

$$(A)\quad 1 + \frac{v}{(1-v)^2} + \frac{v^4}{(1-v)^2(1-v^2)^2} + \frac{v^9}{(1-v)^2(1-v^2)^2(1-v^3)}$$

$$(B)\quad 1 + \frac{v}{1-v} - \frac{v^4}{(1-v)(1-v^2)} + \frac{v^7}{(1-v)(1-v^2)(1-v^3)}$$

and consider determine the nature of the singularities at the points $v = 1$, $v^2 = 1$, $v^3 = 1$, $v^4 = 1$, $v^5 = 1$, ... We know how beautifully the asymptotic nature form of the function can be expressed in a very neat and closed form exponential form. For instance when $v = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} + o(1)^{+}$$

$$(B) = \frac{e^{\frac{\pi^2}{15}t - \frac{t}{60}}}{\sqrt{\frac{2-\sqrt{5}}{2}}} + o(1)^{+}$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $v \to 1$ the function

$$\{(1-v)(1-v^2)(1-v^3)\cdots\}^{120}$$

then is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like A and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $v = e^{-t}$ and $t \to 0$

$$(C)\quad 1 + \frac{v}{(1-v)^2} + \frac{v^3}{(1-v)^2(1-v^2)^2} + \frac{v^6}{(1-v)^2(1-v^2)^2(1-v^3)^2}$$

$$= \sqrt{\left(\frac{t}{2\sqrt{5}}\right)}\, e^{\frac{\pi^2}{6t}} + a_1 t + a_2 t^2 + \cdots + O(a_n t^n)$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on.

The function (2) is a singular function behaving in an (example of 2) simple closed form at the singularities.

* The coeff^{ts} $\pm\frac{1}{n}$ in the index of e happens to be $\frac{\pi^2}{6}$ in this particular case. It may be some other transcendental numbers in other cases.

† The coeff^{ts} of $t, t^2, \ldots$ happen to be $\pm\frac{P}{Q}, \ldots$ in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi\frac{m}{n}}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is:— is the function taken the sum of two functions one of which is an ordinary ϑ function and the other a (finite) function which is $O(1)$ at all the points $e^{2i\pi\frac{m}{n}}$? The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is not inconceivable to construct a ϑ function to cut out the singularities

of the original function. Also (3)
I have shown if it is necessarily
so then it leads to the following
assertion:— viz. it is possible to
construct two power series in x
namely $\sum_0^\infty a_n x^n$ and $\sum_0^\infty b_n x^n$ both
of which have essential singulari-
ties on the unit circle, ~~while~~
~~$\sum_0^\infty a_n x^n$ exists limit, the circle~~
~~and is regular than and $\sum_0^\infty b_n x^n$~~
~~one~~ are convergent when $|x|<1$,
and tend to finite limits at
every point $x = e^{2i\pi r/s}$ and that
at the same time the limit of
of $\sum_0^\infty a_n x^n$ at the point $x = e^{2i\pi r/s}$ is
equal to the limit of $\sum_0^\infty b_n x^n$ at the
point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue.
Anyhow we shall go to the ex-
amples and see how far our as-
sertions are true.

I have proved that if

$$f(v) = 1 + \frac{v}{(1+v)^2} + \frac{v^4}{(1+v)^2(1+v^2)^2} + \cdots$$

then $f(v) + (1-v)(1-v^3)(1-v^5)\cdots(1-2v+2v^4-2v^9+\cdots)$
~~at all the~~ $= O(1)$
at all the points $v = -1$, $v^3 = -1$, $v^5 = -1$,
$v^7 = -1$, ... ;
and at the same time

$$f(v) - (1-v)(1-v^3)(1-v^5)\cdots(1-2v+2v^4-\cdots) = O(1)$$

at all the points $v^2 = -1$, $v^4 = -1$, $v^6 = -1$, ...
Also obviously $f(v) = O(1)$
at all the points $v = 1$, $v^3 = 1$, $v^5 = 1$, ...

And so $f(q)$ is a Mock ϑ function.

Where $q = -e^{-t}$ and $t \to 0$

$$f(q) + \sqrt{\tfrac{\pi}{t}}\, e^{\frac{\pi^2}{2t} - \frac{t}{24}} \to 4.$$

The coeff^t of q^n in $f(q)$ is

$$(-1)^{n-1} \frac{e^{\pi\sqrt{\frac{n}{6}} - \frac{1}{24}}}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{2}{3}}\cdot\frac{1}{24}}}{\sqrt{n-\frac{1}{24}}}\right)$$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

Here are

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \dots$$

These are ~~substantially~~ related to $f(q)$ as shown below.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q)$$

$$= 1 - 2q + 2q^4 - 2q^9 + \dots$$

$$+ \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

Mock ϑ-functions of 5th order

these are of the 2nd order

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q^2)(1+q)} + \frac{q^9}{(1+q^2)(1+q^3)(1+q)} + \dots$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \dots$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \dots$$

$$\chi(q) = 1 + \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \frac{q^3}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$= 1 + \left\{\frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} + \dots\right\}$$

(57)

$$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q^2)(1-q^3)} + \cdots$$

$$\phi(-q) + \chi(q) = 2F(q).$$

$$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$$

$$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)\cdots}$$

$$\psi(q) - F(q^2) + 1 = q \cdot \frac{1 + q^2 + 2q^4 + 2q^6 + q^{12} + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})(1-q^{12})\cdots}$$

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q^2} + \frac{q^6}{(1+q^2)(1+q^4)} + \cdots$$

$$\phi(q) = 1 + q^2 + q^6 + q^{12} + \cdots$$

$$\psi(q) = 1 + \frac{q}{1+q^2} + \frac{q^3}{(1+q^2)(1+q^4)} + \cdots$$

$$\chi(q) = \frac{1-q}{1-q^2} + \frac{(1-q)(1-q^3)}{(1-q^2)(1-q^4)} + \cdots$$

$$F(q) = \frac{1-q}{1-q^2} + \frac{(1-q)(1-q^3)}{(1-q^2)(1-q^4)} + \cdots$$

have got similar relations as above.

Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^2(1-q)}{(1-q^2)(1-q^3)} + \cdots$

(ii) $\frac{1-q}{1-q^2} + \frac{(1-q)(1-q^3)}{(1-q^2)(1-q^4)} + \cdots$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \cdots$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

It is really very remarkable to get a result with an error of $O(n^{-\frac{1}{4}})$ which is as good as that with higher powers of n in this problem. Let C_s be the s-th coefficient which is $O(\sqrt{s})$ and $\Omega(1)$. I suppose now that s is about $\frac{\beta\sqrt{n}}{\log n}$ where $\beta = 4\pi\sqrt{\frac{2}{3}}$. Then $O\sum^{Q(s)} \frac{|C_s|\sqrt{s}}{s\sqrt{n}} \cdot e^{\frac{\pi}{6}\sqrt{\frac{2n}{3}}} = O(s\sqrt{s}\, n^{-\frac{3}{4}}) = O(\log n)^{-\frac{3}{4}}$.

So we would therefore follow from your arguments that the error by taking about $\beta\sqrt{n}/\log n$ terms is $O(\log n)^{-\frac{3}{4}}$. Again suppose that s is about $\frac{\alpha\sqrt{n}}{\log n}$ where $\alpha = \frac{4\pi}{5}\sqrt{\frac{2}{3}}$, Then $\frac{|C_s|\sqrt{s}}{s\sqrt{n}} e^{\frac{\pi}{6}\sqrt{\frac{2n}{3}}} = \Omega\left(\frac{n^{\frac{1}{4}}}{\sqrt{s}}\right) = \Omega(\sqrt{\log n})$.

So therefore appears that in order that $p(n)$ may be the nearest integer to the approximate sum s need not be taken beyond $\beta\sqrt{n}/\log n$ and cannot be taken below $\alpha\sqrt{n}/\log n$ (and at very likely for every n too since the in the above equations appears to be not more negation of a). I hope you can easily prove these. Thus the problem is completely solved. Major Macmahon was kind enough to send me a typewritten copy of the 200 numbers. The approximations gives the exact number. I think you have this already from him. With kind regards to your mother and sister

yours sincerely S. Ramanujan

Properties of $\mathbf{p}(n)$ and $\tau(n)$ defined by the equations

$$\sum_{0}^{\infty} \mathbf{p}(n)x^n = \frac{1}{(1-x)\,(1-x^2)\,(1-x^3)\ldots}$$

$$\sum_{1}^{\infty} \tau(n)x^n = x\,\{(1-x)\,(1-x^2)\,(1-x^3)\ldots\}^{24}$$

Properties of $p(n)$ and $\tau(n)$ defined by the equations

$$\sum_0^\infty p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots},$$

$$\sum_1^\infty \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}.$$

Insert (A)

Modulus 5

1. Let

$$P = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots\right)$$

$$Q = 1 + 240\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right)$$

$$R = 1 - 504\left(\frac{1^5x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \cdots\right)$$

so that

$$Q^3 - R^2 = 1728x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}. \quad (1.1)$$

*. from an elementary proof

Further let J be any function of x with integral coefficients but not the same function throughout, and also let $\sigma_s(n)$ denote the s^{th} powers of the divisors of n. Then it is easy to see that

$$Q = 1 + 5J\ ;\ R = P + 5J. \quad (1.2)$$

Hence

$$Q^3 - R^2 = Q - P^2 + 5J, \quad (1.3)$$

But

$$Q - P^2 = 288\sum_1^\infty n\sigma_7(n)x^n. \quad (1.4)$$

*'

and *'

and it is obvious that

$$\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = \frac{(1-x^{25})(1-x^{50})(1-x^{75})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} + 5J \quad (1.5)$$

It follows from (1.1), (1.3) — (1.5), that

$$x\frac{(1-x^{25})(1-x^{50})(1-x^{75})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_1^\infty n\sigma_7(n)x^n + 5J \quad (1.6)$$

In other words

(1·7) $(1-x^{24})(1-x^{25})(1-x^{50})(1-x^{75})\cdots = \{p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \cdots\}$

$= \sum_1^\infty n\sigma_1(n)x^n + \{J\}$

But the coefficient of x^{5n} on the right-hand side is a multiple of 5. It follows that

(1·8) $p(n-1) \equiv 0 \pmod 5$.

It also follows from (1·7) that

(1·9) $p(n-1) - p(n-26) - p(n-51) + p(n-126) + p(n-176) - p(n-301) - \cdots - n\sigma_1(n) \equiv 0 \pmod 5$,

where $1, 26, 51, 126 \ldots$ are numbers of the form $\frac{1}{2}(5v+1)(15v+2)$ and $\frac{1}{2}(5v-1)(15v-2)$.

The number of values of n for not exceeding 200 for which $p(n) \equiv 0, 1, 2, 3, 4 \pmod 5$ is 64, 33, 34, 34, 30 respectively; and the least value of n for which $p(n) \equiv 4 \pmod 5$ is 30. Thus $p(n) \equiv 4$ appears later than $p(n) \equiv 0 \pmod 5$ for about $\frac{1}{3}$ of the values of n while $p(n) \equiv 1, 2, 3$ or $4 \pmod 5$ for about $\frac{1}{6}$ of the values of n each. It seems extremely difficult to prove any results in this direction concerning $p(n)$, but the problem is much easier concerning $\tau(n)$.

2. It follows from (1·5) and (1·6) that

(2·1) $\begin{cases} \tau(n) - n\sigma_1(n) \equiv 0 \pmod 5, \\ \lambda(n) - n\sigma_1(n) \equiv 0 \pmod 5, \end{cases}$

where $\sum_1^\infty \lambda(n)x^n = x\cdot\dfrac{(1-x^{25})(1-x^{50})(1-x^{75})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}$.

Insert (B)

But if n be written in the form $2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdot 7^{a_7}\cdots$ where the a's are zeros or positive integers then

(2·2) $n\sigma_1(n) = \prod_p \dfrac{p^{a_p}(p^{1+a_p}-1)}{p-1}$, $p = 2, 3, 5, \ldots$

Ric-

(2.3) $$\frac{p^{a_p}(p^{1+a_p}-1)}{p-1} \equiv 0 \pmod 5,$$

if $a_p \geqslant 1$, $p = 5$,

or $a_p \equiv 1 \pmod 2$, $p \equiv 4 \pmod 5$,

or $a_p \equiv 3 \pmod 4$, $p \equiv 2$ or $3 \pmod 5$,

or $a_p \equiv 4 \pmod 5$, $p \equiv 1 \pmod 5$,

and for no other values. Suppose now that

(2.4) $$\begin{cases} t_n = 0, & \tau(n) \equiv 0 \pmod 5 \\ t_n = 1, & \tau(n) \not\equiv 0 \pmod 5. \end{cases}$$

Then it follows from (2.3) that

(2.5) $$\sum_1^\infty \frac{t_n}{n^s} = \Pi_1 \Pi_2 \Pi_3$$

where $$\Pi_1 = \prod_p \frac{1}{1-p^{-2s}},$$

p being a prime of the form $5k-1$, and

$$\Pi_2 = \prod_p \frac{1-p^{-3s}}{(1-p^{-s})(1-p^{-4s})}$$

p being a prime of the form $5k \pm 2$ and

$$\Pi_3 = \prod_p \frac{1-p^{-4s}}{(1-p^{-s})(1-p^{-5s})}$$

p being a prime of the form $5k+1$.

It is easy to prove from (2.5) that

(2.6) $$t_1 + t_2 + t_3 + \cdots + t_n = o(n).$$

It can be shown by transcendental methods that

(2.7) $$t_1 + t_2 + t_3 + \cdots + t_n \sim \frac{C n}{(\log n)^{\frac{1}{2}}},$$

and

(2.8) $$t_1 + t_2 + t_3 + \cdots + t_n = C \int_1^n \frac{dx}{(\log x)^{\frac{1}{2}}} + O\left(\frac{n}{(\log n)^R}\right),$$

where C is a constant and R is any positive number.

Add. to a 94 (3/14) (4)

The proof of (2.6) is quite elementary and very similar to that for showing that $\pi(x) = o(x)$ *

* See Landau's Vorlesungen p

$\pi(x)$ being the number of primes not exceeding x. The results (2.6) can be stated roughly in other words that $\tau(n)$ and $\lambda(n)$ are ~~almost always~~ divisible by 5 for almost all values of n, while (2.7) and (2.8) give a lot more information.

Modulus 25

3. It is easily seen from (1.2) that

$$Q^5 - R^2 = 2(Q^2 - PR) - (Q - P^2) + Q(Q-1)^2 - (R-P)^2$$
$$= 2(Q^2 - PR) - (Q - P^2) + 25\,\mathrm{J}. \tag{3.1}$$

But

$$Q^2 - PR = 1008 \sum_1^\infty n\,\sigma_5(n)\,x^n \; ; \tag{3.2}$$

and it is obvious that

$$\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^{24} = \frac{\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^5}{(1-x)(1-x^2)(1-x^3)\cdots} + 25\,\mathrm{J}. \tag{3.3}$$

Now remembering that

$$\sigma_5(n) - \sigma_1(n) \equiv 0 \pmod 5 \tag{3.4}$$

it follows from (1.4) and (3.1) — (3.3) that

$$x\frac{\left\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\right\}^5}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_1^\infty \left\{2n\sigma_5(n) - n\sigma_1(n)\right\} x^n + 25\,\mathrm{J} \tag{3.5}$$

From this it is easily deduced that

$$\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^5 \left\{p(4)x + p(9)x^2 + p(14)x^3 + \cdots\right\}$$
$$= \sum_1^\infty \left\{10n\sigma_5(n) - 5n\sigma_1(n)\right\} x^n + 25\,\mathrm{J},$$

and hence that

$$(1-x^5)(1-x^{10})(1-x^{15})\cdots\left\{p(4)x + p(9)x^2 + p(14)x^3 + \cdots\right\}$$
$$\equiv 5\sum_1^\infty n\sigma_1(n)x^n + 25\,\mathrm{J}. \tag{3.6}$$

Since the coefficient of x^{5n} is a multiple

of 25ⁿ it follows that

(3.7) $$p(25n-1) \equiv 0 \pmod{25}$$

It also follows from (3.6) that

(3.8) $$p(5n-1) + p(5n-26) - p(5n-51) + p(5n-126) + p(5n-176) - \cdots - 5n\,\sigma_7(n) \equiv 0 \pmod{25},$$

where 1, 26, 51, 126, ... are the same as in (1.7).

4. It is easy to see that

(4.1) $$n\,\sigma_9(n) - 2n\,\sigma_7(n) + n\,\sigma_5(n) \equiv 0 \pmod{25}.$$

It follows from this and (3.5) and (3.5) that

(4.2) $$\tau(n) - n\,\sigma_9(n) \equiv 0 \pmod{25}.$$

It appears that, if k be any positive integer, it is possible to find two integers a and b such that

(4.3) $$\tau(n) - n^a\,\sigma_b(n) \equiv 0 \pmod{5^k},$$

Thus for instance if n is not a multiple of 5

(4.4) $$\tau(n) - n^{41}\,\sigma_{29}(n) \equiv 0 \pmod{125},$$

I have not yet proved these results.

It also appears that the coefficient of x^n in the left hand side of (3.5) can be exactly determined in terms of the real divisors of n. Thus

Insert (C)

(4.5) $$x\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{(1-x)(1-x^2)(1-x^3)\cdots} = \frac{x}{(1-x)^2} - \frac{x^2}{(1-x^2)^2} - \frac{x^3}{(1-x^3)^2} + \frac{x^4}{(1-x^4)^2} + \frac{x^6}{(1-x^6)^2} - \frac{x^7}{(1-x^7)^2} - \cdots$$

The allied function

(4.6) $$\frac{\{(1-x)(1-x^2)(1-x^3)\cdots\}^5}{(1-x^5)(1-x^{10})(1-x^{15})\cdots} = 1 - 5\left(\frac{x}{1-x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \frac{7x^7}{1-x^7} - \cdots\right).$$

It follows from (4.5) that

$$\{(1-x)(1-x^2)(1-x^3)\cdots\}^5\{p(1)x + p(2)x^2 + p(3)x^3 + \cdots\}$$
$$= 5\left\{\frac{x}{(1-x)^2} - \frac{x^2}{(1-x^2)^2} - \frac{x^3}{(1-x^3)^2} + \frac{x^4}{(1-x^4)^2} + \frac{x^6}{(1-x^6)^2} - \cdots\right\}$$

Add this as 34.13(c) (1.) (11)

and hence that

(4.7) $$p(4) + p(9)x + p(14)x^2 + p(19)x^3 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6}$$ *

* For a direct proof of this result see §

Modulus 7

5. Since

(5.1) $$Q^2 = 1 + 480\left(\frac{1^7x}{1-x} + \frac{2^7x^2}{1-x^2} + \frac{3^7x^3}{1-x^3} + \cdots\right)$$ *1

it is easy to see that

(5.2) $$Q^7 = P + 7J; \quad R = 1 + 7J;$$

(5.3) and so $$(Q^3 - R^2)^2 \equiv P^3 - 2PQ + R + 7J.$$

But

(5.4) $$\left\{\begin{aligned} PQ - R &= 720\sum_1^\infty \sigma_3(n)x^n \cdot * \\ P^3 - 2PQ + 2R &= -1728\sum_1^\infty n^2\sigma(n)x^n; \end{aligned}\right.$$

and it is obvious that

(5.5) $$\{(1-x)(1-x^2)(1-x^3)\cdots\}^{48} = \frac{(1-x^{49})(1-x^{98})(1-x^{147})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} + 7J.$$

It follows from all these that

(5.6) $$x^2\frac{(1-x^{49})(1-x^{98})(1-x^{147})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_1^\infty\{n^3\sigma_1(n) - n\sigma_3(n)\}x^n + 7J.$$

In other words

(5.7) $$(1-x^{49})(1-x^{98})(1-x^{147})\cdots\{p(0)x^2 + p(1)x^3 + p(2)x^4 + \cdots\} = \sum_1^\infty\{n^3\sigma_1(n) - n\sigma_3(n)\}x^n + 7J.$$

It follows that

(5.8) $$p(7n-2) \equiv 0 \pmod 7,$$

and

(5.9) $$p(n-2) - p(n-51) - p(n-100) + p(n-247) + p(n-345) - \cdots + n\sigma_3(n) - n^3\sigma_1(n) \equiv 0 \pmod 7$$

where 2, 51, 100, 247, ... are numbers of the form $\frac{1}{2}(7\nu+1)(3\nu+4)$ and $\frac{1}{2}(7\nu-1)(3\nu-$

5. The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7$ is 50, 33, 22, 28, 23, 23, 21 respectively; and the least value of n for which $p(n) \equiv 6 \pmod 7$ is 73. It appears that $p(n) \equiv 0 \pmod 7$ for about $\frac{1}{4}$ of the values of n while $p(n) \equiv 1, 2, 3, 4, 5, 6 \pmod 7$ for about $\frac{1}{8}$ of the values of n each.

6. It follows from (5·2) that

$$Q^3 - R^2 = PQ\,R + 7J. \tag{6·1}$$

It is easy to see from this and § 4 that

$$\tau(n) - n\sigma_3(n) \equiv 0 \pmod 7. \tag{6·2}$$

Now if $n = 2^{a_2} 3^{a_3} 5^{a_5} \dots p^{a_p} \dots$ then

$$n\sigma_3(n) = \prod_p p^{a_p} \frac{p^{3(1+a_p)} - 1}{p^3 - 1}, \quad p = 2, 3, 5, 7, \dots \tag{6·3}$$

But

$$p^{a_p} \frac{p^{3(1+a_p)} - 1}{p - 1} \equiv 0 \pmod 7 \tag{6·4}$$

if $a_p \equiv 6 \pmod 7$, $p \equiv 1, 2, 4 \pmod 7$,

or $a_p \equiv 1 \pmod 2$, $p \equiv 3, 5, 6 \pmod 7$

or $a_p \geqslant 1$, $p = 7$.

Suppose now that

$$t_n = 1, \quad \tau(n) \not\equiv 0 \pmod 7,$$
$$t_n = 0, \quad \tau(n) \equiv 0 \pmod 7.$$

Then it follows from (6·4) that

$$\sum_1^\infty \frac{t_n}{n^s} = \Pi_1 \Pi_2 \tag{6·5}$$

where $\Pi_1 = \prod_p \dots$ p being a prime of the form $7k+1$, $7k+2$, $7k+4$, and

$$\Pi_2 = \prod_p \frac{1}{1 - p^{-7s}}$$

p being a prime of the form $7k+3$, $7k+5$, $7k+6$. It is easy to prove (by quite

(from (6·5))

elementary methods that

$$t_1 + t_2 + t_3 + \cdots + t_n = o(n), \tag{6·6}$$

It can be shown by Tauberian methods that

$$t_1 + t_2 + t_3 + \cdots + t_n \sim \frac{C n}{(\log n)^{\frac{1}{2}}};$$

and

$$t_1 + t_2 + t_3 + \cdots + t_n = C \int_1^n \frac{dx}{(\log x)^{\frac{1}{2}}} + O\frac{n}{(\log n)^k} \tag{6·7}$$

where k is any positive number and

$$C = \frac{6^{\frac{1}{2}}}{7^{\frac{3}{2}}} \cdot \frac{1-2^{-6}}{1-2^{-7}} \cdot \frac{1-11^{-6}}{1-11^{-7}} \cdot \frac{1-23^{-6}}{1-23^{-7}} \cdot \frac{1-29^{-6}}{1-29^{-7}} \cdots$$

$$\times \left\{ \frac{1}{(1-3^{-2})(1-5^{-2})(1-13^{-2})(1-17^{-2})(1-19^{-2})\cdots} \right\}^{\frac{1}{2}},$$

$2, 11, 23, \cdots$ being primes of the form $7k+1$, $7k+2$ and $7k+4$ while $3, 5, 13, \cdots$ being primes of the form $7k+3$, $7k+5$ and $7k+6$.

Thus we see that $\tau(n)$ is divisible by 7 for almost all values of n; and at the same time the number of values of n for which $\tau(n)$ is divisible by 7 is far more numerous than that for which $\tau(n)$ is divisible by 5. Now if

$$\sum_1^\infty \lambda(n) x^n = x \frac{(1-x^{49})(1-x^{98})(1-x^{147})\cdots}{(1-x)^7 (1-x^2)^7 (1-x^3)^7 \cdots}$$

it is clear from (5·6) that Insert (P)

$$\lambda(n) - n^2 \sigma_1(n) + n\sigma_3(n) \equiv 0 \pmod 7. \tag{6·8}$$

But it is easy to show that $n^2\sigma_1(n)$ and $n\sigma_3(n)$ are divisible by 7 for almost all values of n. It follows that $\lambda(n)$ is divisible by 7 for almost all values of n. It can then be shown that the number of values of n not exceeding n for which $\lambda(n)$ is not divisible by n is

$$O\frac{n}{(\log n)^k}. \tag{6·9}$$

The index $\frac{1}{2}$ in (6·9) is easily obtained by considering $n^2\sigma_1(n)$ and $n\sigma_3(n)$ separately; but whether this is the right index or not can be known only by considering

$$n^2\sigma_7(n) - n\,\sigma_3(n) \qquad (9)$$

to be taken together, which seems rather complicated to deal with.

Modulus 49

7. We have

$$(Q^3-R^2)^2 = (3P^2Q^2 - 4PQR - 2Q^3 + 3R^2) - 2(P^3 - 2PQ + R) + 49J$$

in virtue of (5·2). But

$$(7.1)\quad \begin{cases} Q^3 - R^2 = 1728 \sum_1^\infty \tau(n)\,x^n, \\ 3Q^3 + 2R^2 - 5PQR = 1584 \sum_1^\infty n\,\sigma_3(n)\,x^n, \\ 5Q^3 + 4R^2 - 18PQR + 9P^2Q^2 = 8640 \sum_1^\infty n^2\sigma_7(n)\,x^n; \end{cases}$$

and it is obvious that

$$(7.2)\quad \{(1-x)(1-x^2)(1-x^3)\cdots\}^{48} = \frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{(1-x)^7(1-x^2)^7(1-x^3)^7\cdots} + 49J.$$

Now remembering that

$$(7.3)\quad \begin{cases} \sigma_7(n) - \sigma(n) \equiv 0 \pmod 7, \\ \sigma_9(n) - \sigma_3(n) \equiv 0 \pmod 7 \end{cases}$$

it follows from the above equations and (5·1) that

$$(7.4)\quad x^2\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{(1-x)^7(1-x^2)^7(1-x^3)^7\cdots} = \sum_1^\infty \{2n\sigma_7(n) - 4n^2\sigma_7(n) + 2n\sigma_3(n) - 2n^2\sigma_7(n) + 2\tau(n)\}x^n + 49J,$$

From this we deduce that

$$(7.5)\quad \{(1-x)(1-x^2)(1-x^3)\cdots\}^7 \{p(1)x + p(2)x^2 + p(3)x^3 + \cdots\} = \sum_1^\infty \{28n\sigma_7(n) + 2\tau(n)\}x^n + 49J.$$

I have stated in my previous paper that

$$(7.6)\quad \sum_1^\infty \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}} \qquad *$$

(10)

* which p assumes all prime values. This has since been proved by Mr Mordell.

Now by actual calculation we find that

$$\tau(7) \equiv 14 \pmod{49}.$$

It follows from this and (7.6) that

$$\tau(7n) - 14\tau(n) \equiv 0 \pmod{49}.$$

It is easy to see from this and (7.5) that

(7.7) $$(1-x^7)(1-x^{14})(1-x^{21})\cdots\{p(5)x + p(12)x^2 + p(19)x^3 + \cdots\} = 7\sum_1^\infty n\sigma_3(n)x^n + 49J$$

Now if $n \equiv 3, 5, 6 \pmod 7$

then n must contain an odd power of a prime of the form $7k+3$, $7k+5$ or $7k+6$ as a divisor since all perfect squares are of the form $7k$, $7k+1$, $7k+2$ or $7k+4$, and so $\sigma_3 n$ is divisible by p^3+1 which is divisible by 7. Also it is obvious that if n is a multiple of 7 then $n\sigma_3(n)$ is also divisible by 7. It follows that if

$$n \equiv 0, 3, 5, 6 \pmod 7$$

then $$n\sigma_3(n) \equiv 0 \pmod 7.$$

It is easy to see from this and (7.7) that

(7.8) $$\left.\begin{matrix} p(49n-2), \\ p(49n-9), \\ p(49n-16), \\ p(49n-30) \end{matrix}\right\} \equiv 0 \pmod{49}$$

It also follows from (7.7) that

(7.9) $$p(7n-2) - p(7n-51) - p(7n-100) + p(7n-247) + p(7n-345) - \cdots - 7n\sigma_3(n) \equiv 0 \pmod{49}$$

where 2, 51, 100, 247, ... are the same as in (5.9).

Add to a 12(11) (11)

8. It appears that

$$(8.1)\quad x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3 + 8x^2\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{(1-x)(1-x^2)(1-x^3)\cdots}$$
$$= x\frac{1+x}{(1-x)^3} + x^2\frac{1+x^2}{(1-x^2)^3} - x^3\frac{1+x^3}{(1-x^3)^3} + x^4\frac{1+x^4}{(1-x^4)^3} - x^5\frac{1+x^5}{(1-x^5)^3} - x^6\frac{1+x^6}{(1-x^6)^3} + x^8\frac{1+x^8}{(1-x^8)^3} + \cdots;$$

while the allied function

$$(8.2)\quad 49x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3 + 8\frac{\{(1-x)(1-x^2)(1-x^3)\cdots\}^7}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}$$
$$= 8 - 7\left(\frac{x}{1-x} - \frac{4x^2}{1-x^2} - \frac{9x^3}{1-x^3} + \frac{16x^6}{1-x^4} - \frac{25x^5}{1-x^5} - \frac{36x^6}{1-x^6} + \frac{64x^8}{1-x^8} + \cdots\right)$$

Now remembering that

$$\{(1-x)(1-x^2)\cdots\}^3 = 1 - 3x + 5x^3 - 7x^6 + \cdots$$

and picking out the terms $x^7, x^{14}, x^{21}, \ldots$ from both sides in (8.1) we obtain

$$-7x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3 + 8\{(1-x)(1-x^2)(1-x^3)\cdots\}^7\{p(7)x^7 + p(14)x^{14} + p(21)x^{21} + \cdots\}$$
$$= 49\{x\frac{1+x}{(1-x)^3} + x^2\frac{1+x^2}{(1-x^2)^3} - x^3\frac{1+x^3}{(1-x^3)^3} + \cdots\}$$

the series in the right hand side being the same as that in (8.1). It follows from this and (8.1) that

$$(8.3)\quad p(5) + p(12)x + p(19)x^2 + p(26)x^3 + \cdots = 7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^4} + 49x\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^8}$$

* From a direct proof of this see §

(1.2)

It also appears that if

$$\sum_1^\infty \lambda(n) x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3 \{(1-x^2)(1-x^4)\cdots\}^3 / \{(1-x^{2+1})\cdots\}^3$$

then

$$(8.4)\qquad \sum_1^\infty \frac{\lambda(n)}{n^s} = \frac{1}{1+7^{1-s}}\,\Pi_1\,\Pi_2,$$

where

$$\Pi_1 = \prod_p \frac{1}{1-p^{2-2s}},$$

p being a prime of the form $7k+3, 7k+5$, or $7k+6$, and

$$\Pi_2 = \prod_p \frac{1}{1+(2p-a^2)p^{-s}+p^{2-2s}}$$

p being a prime of the form $7k+1, 7k+2$ or $7k+4$ and a and b being integers such that $4p = a^2+7b^2$. Thus $\lambda(n)$ can be completely ascertained. It follows from this and (8.1) and (8.3) that the coefficients of x^n in

$$\frac{\{(1-x)(1-x^2)(1-x^3)\cdots\}^7}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}, \qquad x^2\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^7}{(1-x)(1-x^2)(1-x^3)\cdots}$$

can be completely ascertained.

Now it is easy to see that

$$3n^2 - 2n^3 \equiv 0, 1, -1 \pmod{49}$$

according as $n \equiv 0 \pmod 7$, $n \equiv 1, 2, 4 \pmod 7$ or $n \equiv 3, 5, 6 \pmod 7$. Also the coefficient of x^n in $x(1+x)/(1-x)^3$ is n^2. Hence the right hand side in (8.1) can be written as

$$(8.5)\qquad \sum_1^\infty \{3n^2\sigma_7(n) - 2n^3\sigma_1(n)\}x^n + 49\,J,$$

It follows from this and (7.4) and (8.1) that

$$(8.6)\qquad \tau(n) - 3\lambda(n) + n\sigma_7(n) + n\sigma_3(n) \equiv 0 \pmod{49},$$

Add to (8.4) (13)

where $\lambda(n)$ is the same as in (8.4). From the formulae (8.4) and (8.6) all the residues of $\tau(n)$ for moduli 49 can be completely ascertained.

Modulus 11

9. In this case we start with the series

$$(9.1)\quad \begin{cases} 1 - 264\left(\frac{19x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \cdots\right) = QR, \\ 691 + 65520\left(\frac{1^{11}x}{1-x} + \frac{2^{11}x^2}{1-x^2} + \frac{3^{11}x^3}{1-x^3} + \cdots\right) = 441Q^2 + 250R^3. \end{cases}$$

It follows that

$$(9.2)\quad QR = 1 + 11J\,; \quad Q^3 - 3R^2 = -2P + 11J$$

It is easy to see from this that

$$(Q^3 - R^2)^5 = (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 - R(Q^3 - 3R^2)^2$$
$$- 5QR + 11J = P^5 - 3P^3Q - 4P^2R - 5QR + 11J.$$

But

$$(9.3)\quad \begin{cases} P^5 - 10P^3Q + 20P^2R - 15PQ^2 + 4QR = -20736 \sum_1^\infty n^4\sigma_7(n)x^n, \\ P^3Q - 3P^2R + 3PQ^2 - QR = 3456 \sum_1^\infty n^3\sigma_3(n)x^n, \\ P^2R - 2PQ^2 + QR = -1728 \sum_1^\infty n^2\sigma_3(n)x^n, \\ PQ^2 - QR = 720 \sum_1^\infty n\,\sigma_7(n)x^n; \end{cases}$$

and

and it is obvious that

$$(9.4)\quad \left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{120} = \frac{(1-x^{121})(1-x^{242})(1-x^{363})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} + 11J$$

It is easy to see from all these that

$$(9.5)\quad x^5\frac{(1-x^{121})(1-x^{242})(1-x^{363})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} =$$
$$\sum_1^\infty \left\{-n^4\sigma_7(n) + 3n^3\sigma_3(n) - 5n\sigma_7(n)\right\}x^n + 11J.$$

It follows from this that

$$(9.6)\quad p(11n-5) \equiv 0 \pmod{11};$$

and

$$(9.7)\quad p(n-5)-p(n-126)-p(n-247)+p(n-610)+p(n-852)-\cdots+n^4\sigma_7(n)-3n^3\sigma_5(n)-3n^2\sigma_5(n)+5n\sigma_7(n)\equiv 0 \pmod{11},$$

It is only to prove the general result (9.7) we require all the details in (9.3). But we don't require all these details in order to prove (9.6) and the proof can be very much simplified as follows: we have

$$(9.8)\quad \begin{cases} x\dfrac{dP}{dx}=\dfrac{P^2-Q}{12} \\ x\dfrac{dQ}{dx}=\dfrac{PQ-R}{3} \\ x\dfrac{dR}{dx}=\dfrac{PR-Q}{2} \end{cases}$$

Now using (9.2) and (9.8) we can show that

$$(Q^3-R^2)^5 = x\frac{dJ}{dx}+11J$$

As mentioned in the beginning, the J's are not the same functions.

It follows from this and (9.4) that

$$(9.9)\quad x^5\cdot\frac{(1-x^{121})(1-x^{242})(1-x^{363})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}=x\frac{dJ}{dx}+11J.$$

Since the coefficient of x^{11n} in the right-hand side is a multiple of 11 it follows that

$$p(11n-5)\equiv 0 \pmod{11}.$$

where 5, 126, 247, 610, ... are numbers of the form $\frac{1}{2}(11\nu+2)(33\nu+5)$ and $\frac{1}{2}(11\nu-2)(33\nu-5)$.

The number of values of n not exceeding 201 for which $p(n)\equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11}$ is 77, 23, 24, 14, 15, 14, 5, 12, 8, 8, 0 respectively.

Add 1 : 2 9 11 13 (15) (15)

Even though these values seem to be very irregular it appears from the residues of $p(n)$ for moduli 5 and 7 and also from the next section that $p(n) \equiv 0 \pmod{11}$ for about $\frac{1}{6}$ of the values of n while $p(n) \equiv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11}$ for about $\frac{1}{12}$ of the values of n each.

10. Mr H.B.C. Darling observed the remarkable fact (before I began to write this paper) that $p(n)$ is divisible by 11 for 45 values of n not exceeding 100. This can be explained by the formula (9.7) and the congruency of

$$n^4\sigma_1(n) - 3n^3\sigma_3(n) - 3n^2\sigma_5(n) + 5n\sigma_7(n) \tag{10.1}$$

for modulus 11. It can be shown by quite elementary method that (10.1) is divisible by 11 for almost all values of n. It can even be shown that the number of values of n not exceeding n for which (10.1) is not divisible by 11 is

$$O\frac{n}{(\log n)^{\frac{1}{10}}} \tag{10.2}$$

by considering the divisibility of the four terms in (10.1) separately; but a better result can be found only by considering all the four terms in (10.1) taken together. The same remarks apply to the function $\lambda(n)$ defined by

$$\sum_1^\infty \lambda(n)\, x^n = x^{\frac{1}{24}}\frac{(1-x^{121})(1-x^{242})(1-x^{363})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}; \tag{10.3}$$

that is to say $\lambda(n)$ is divisible by 11 for

Add to p. 13 (16)

almost all values of n; and the number of values of $\lambda(n)$, not divisible by 11 is of the form (10.7). It appears that the number of values of n for which $p(n) \equiv 0 \pmod{11}$ (from (10.7)) cannot be so high as 45% if n exceeds 120. Thus the number of values of n divisible by 11 is

45%, $0 < n \leq 40$

45%, $40 < n \leq 80$

45% $80 < n \leq 120$

35%, $120 < n \leq 160$

$22\frac{1}{2}$%, $160 < n \leq 200$.

It is also very remarkable that, in the table of the first 200 values of $p(n)$, there is not a single value of $p(n)$ of the form $11k-1$. This is probably due to such a high percentage of the values of $p(n)$ divisible by 11 in the beginning.

I have not yet investigated completely the residues of $\tau(n)$ for modulus 11. But it appears that if

$$\sum_1^\infty \lambda(n) x^n = x\{(1-x)(1-x^2)\cdots\}^2\{(1-x^{11})(1-x^{22})\cdots\}^2$$

then

$$\sum_1^\infty \frac{\lambda(n)}{n^s} = \frac{1}{1-11^{-s}} \prod_p \frac{1}{1-\lambda(p)p^{-s}+p^{1-2s}} \tag{10.4}$$

p being assuming all prime values except 11 and that $\lambda(p)$ can be determined also. If that is so then the residues of $\tau(n)$ for modulus 11 can also be ascertained since it is easily seen that (Insert F)

$$\tau(n) - \lambda(n) \equiv 0 \pmod{11} \tag{10.5}$$

(17)

Moduli 2 and 3

11. Before we proceed to consider higher moduli we shall see what the analogous formulae are in the cases of moduli 2 and 3. It is easy to see that

$$(11.1)\qquad \frac{(1-x^2)(1-x^8)(1-x^{12})\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = \frac{(1-x^3)(1-x^6)(1-x^9)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} + 2J = 1+x+x^3+x^6+\cdots+2J.$$

It follows that

$$(11.2)\qquad p(n)-p(n-4)-p(n-8)+p(n-20)+p(n-28)-\cdots$$

is odd or even according as n is a triangular number or not, 4, 8, 20, ... being numbers of the form $2V(3V+1)$ and $2V(3V-1)$.

$p(n)$ is odd for 110 values of n not exceeding 200 and even for 90 values of n in the same range. Thus $p(n)$ seems to be odd for more values of n than those for which $p(n)$ is even.

Again we have

$$(11.3)\qquad \frac{(1-x^9)(1-x^{18})(1-x^{27})\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = \frac{\{(1-x^3)^3(1-x^6)^3(1-x^9)^3\cdots\}^3}{(1-x)(1-x^2)(1-x^3)\cdots} + 3J$$

But it can be shown that [Insert (H)]

$$(11.4)\qquad x\frac{\{(1-x^9)(1-x^{18})(1-x^{27})\cdots\}^3}{(1-x^3)(1-x^6)(1-x^9)\cdots} = \frac{x}{1+x+x^3}+\frac{x^2}{1+x^2+x^4}+\frac{x^4}{1+x^4+x^8}+\frac{x^5}{1+x^5+x^{10}}+\frac{x^7}{1+x^7+x^{14}}+\cdots.$$

But the right hand side in (11.4) is of the form

$$\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\frac{x^4}{(1-x^4)^2}+\frac{x^5}{(1-x^5)^2}+\cdots+3J;$$

and (11.2) and the coefficient of x^{3n+1} in the above series is $\sigma_1(3n+1)$. It follows from this and (11.4) that

$$(11.5)\qquad \frac{(1-x^9)(1-x^{18})(1-x^{27})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_0^\infty \sigma_1(3n+1)x^n + 3J.$$

From this we easily deduce that

(18)

(11.6) $p(n) - p(n-9) - p(n-18) + p(n-45) + p(n-63)$
$- p(n-108) - \cdots - \sigma_1(3n+1) \equiv 0 \pmod 3$,
where 9, 18, 45, ... are numbers of the form
$\frac{3}{2}\nu(3\nu+1)$ and $\frac{3}{2}\nu(3\nu-1)$. (not exceeding 200)

The number of values of n for which
$p(n) \equiv 0, 1, 2 \pmod 3$ is 66, 68, 66 respectively.
Thus it appears that $p(n) \equiv 0, 1, 2 \pmod 3$ for
about $\frac{1}{3}$ of the number of values of n exist.

Function properties of $\tau(n)$ Insert [G]

12. It is easy to see that
$$\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^{24} = \left\{\frac{(1-x^2)(1-x^4)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots}\right\}^8$$
$$+ 32J = (1+x+x^3+x^6+\cdots)^8 + 32J.$$
But
$$x(1+x+x^3+x^6+\cdots)^8 = \frac{1^3x}{1-x^8} + \frac{2^3x^2}{1-x^{16}} + \frac{3^3x^3}{1-x^{24}} + \cdots;$$
and $1^4x + 2^4x^2 + 3^4x^3 + \cdots = \frac{x}{1-x^8} + 16J$,
and $1^8x + 2^8x^2 + 3^8x^3 + \cdots = \frac{x}{1-x^8} + 32J$.
It is easy to see from all these that

(12.1) and (12.2)
$$\begin{cases} \tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{16}; \\ \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{32}. \end{cases}$$

Again we have
$$\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^{24} = \frac{\left\{(1-x^3)(1-x^6)(1-x^9)\cdots\right\}^9}{\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^3}$$
$+ 97J$
But it can be shown that

2/ (12.3)
$$x\frac{\left\{(1-x^3)(1-x^6)(1-x^9)\cdots\right\}^9}{\left\{(1-x)(1-x^2)(1-x^3)\cdots\right\}^3} = \frac{1^3x}{1+x+x^2} + \frac{2^3x^2}{1+x^2+x^4} + \frac{3^3x^3}{1+x^3+x^6} + \cdots$$
Now it is easy to see that
$$1^3x + 2^3x^2 + 3^3x^3 + \cdots = \frac{x}{1+x+x^2} + 9J,$$
and
$$1^9x + 2^9x^2 + 3^9x^3 + \cdots = \frac{x}{1+x+x^2} + 27J$$

(19)

It follows that

$$(12.3)\quad \begin{cases} \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod 9, \\ \tau(n) - n^2\sigma_3(n) \equiv 0 \pmod{27}. \end{cases}$$

It is easy to deduce from (9.1), (6.2), (6.8), (12.1) and (12.3) that

$$(12.4)\quad \begin{cases} \tau(n) - n\sigma_1(n) \equiv 0 \pmod{30}, \\ \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod{36}, \\ \tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{48}, \\ \tau(n) - n^5\sigma_1(n) \equiv 0 \pmod{120}, \end{cases}$$

$$(12.5)\quad \begin{cases} \tau(n) - n\sigma_3(n) \equiv 0 \pmod{42}, \\ \tau(n) - n^2\sigma_3(n) \equiv 0 \pmod{60}, \\ \tau(n) - n^4\sigma_3(n) \equiv 0 \pmod{168}, \end{cases}$$

$$(12.6)\quad \begin{cases} \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{288}, \\ \tau(n) - n^2\sigma_7(n) \equiv 0 \pmod{540}, \\ \tau(n) - n\sigma_9(n) \equiv 0 \pmod{1050}. \end{cases}$$

Again it easily follows from the second equation in (9.1) that

$$(12.7)\quad \tau(n) - \sigma_{11}(n) \equiv 0 \pmod{691}.$$

Insert (ω^3) & (ω)

Modulus 13

13. In this case we start with the second series in (9.1) and the series

$$(13.1)\quad 1 - 24\left(\frac{1^{13}x}{1-x} + \frac{2^{13}x^2}{1-x^2} + \frac{3^{13}x^3}{1-x^3} + \cdots\right) = Q^3R$$

It follows from these that

$$(13.2)\quad Q^3 - 3R^2 = -2 + 13J\,; \quad Q^3R = P + 13J.$$

Hence we have

$$(Q^3-R^2)^7 = -2(R^2-1)^7 + 13J = -5R^2(3R^2-2)^4 - 2R^4(3R^2-2)^3 + 6R^4(3R^2-2)^2 - 6R^2(3R^2-2)^2 - 6R^2(3R^2-2) - 2(R^2-1) + 13J = -5P^6 - 2P^4Q + 6P^3R - 6P^2Q^2 - 6PQR - (Q^3-R^2) + 13J.$$

But

(20)

$$
(13.3)\quad
\begin{cases}
5(P^6 - 15P^4Q + 40P^3R - 45P^2Q^2 + 24PQR)^2 \\
\quad - (9Q^3 + 16R^2)^2 = -2488320 \sum_1^\infty n^5\sigma_1(n)x^n, \\
7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + (3Q^2 + 4R^2) \\
\quad = 41472 \sum_1^\infty n^4\sigma_3(n)x^n, \\
2(P^3R - 3P^2Q^2 + 3PQR) - (Q^3 + R^2) \\
\quad = -5184 \sum_1^\infty n^3\sigma_5(n)x^n, \\
9(PQ - R)^2 + 5(Q^3 - R^3) = 8640 \sum_1^\infty n^3\sigma_7(n)x^n, \\
5PQR - (Q^3 + R^2) = -1584 \sum_1^\infty n\sigma_9(n)x^n, \\
Q^3 - R^2 = 1728 \sum_1^\infty \tau(n)x^n;
\end{cases}
$$

and it is obvious that

$$
(13.4)\quad \{(1-x)(1-x^2)(1-x^3)\cdots\}^{168}
= \frac{(1-x^{169})(1-x^{338})(1-x^{507})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} + 13\mathfrak{J}.
$$

It is easy to see from all these that

$$
(13.5)\quad x^7\frac{(1-x^{169})(1-x^{338})(1-x^{507})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}
= (1-x^{169})(1-x^{338})(1-x^{507})\cdots\{p(0)x^7 + p(1)x^8 + p(2)x^9 + p(3)x^{10} + \cdots\}
$$

$$
= \sum_1^\infty \{n^5\sigma_1(n) - 4n^4\sigma_3(n) - 3n^3\sigma_5(n) + 6n^2\sigma_7(n) - 3n\sigma_9(n) + 3\tau(n)\}x^n + 13\mathfrak{J},
$$

It is easy to see by actual calculation that $\tau(13) \equiv 8 \pmod{13}$. It follows from this and (13.5) that

$$
(13.6)\quad \{p(0)x + p(13)x^2 + p(26)x^3 + p(39)x^4 + \cdots\}(1-x^{13})(1-x^{26})(1-x^{39})\cdots
= 11\sum_1^\infty \tau(n)x^n + 13\mathfrak{J}.
$$

It is not necessary to know all the details above in order to prove (13.6). The proof can be very much simplified as follows: (in virtue of (7.6) and hence $\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}$)

using (9.8) and (13.2) we can show that (21)

(13.7) $$(Q^3-R^2)^7 = \frac{x\,dJ}{dx} + 3(Q^3-R^2) + 13J.$$

It follows from this that

(13.8) $$x^7\frac{(1-x^{169})(1-x^{338})(1-x^{507})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \frac{x\,dJ}{dx} + 3\sum_1^\infty \tau(n)x^n + 13J.$$

From this we easily deduce (13.6).

Again picking out the terms $x^{13}, x^{26}, x^{39}, \ldots$ in (13.6) we obtain

(13.9) $$\{p(13^2-7)x + p(2\cdot 13^2-7)x^2 + p(3\cdot 13^2-7)x^3 + p(4\cdot 13^2-7)x^4 + \ldots\}(1-x)(1-x^2)(1-x^3)\cdots = 10\sum_1^\infty \tau(n)x^n + 13J.$$

14. The formulae (13.6) and (13.9) can be written as

(14.1) $$\sum_0^\infty p(13n+6)x^n = 11\{(1-x)(1-x^2)(1-x^3)\cdots\}^{11} + 13J;$$

and

(14.2) $$\sum_0^\infty p(13^2n+163)x^n = 23\{(1-x)(1-x^2)(1-x^3)\cdots\}^{23} + 13J.$$

Since I began to write this paper I have found by a different method that if λ be any positive odd integer then

(14.3) $$\sum_0^\infty p\left(13^\lambda n + \frac{11\cdot 13^\lambda+1}{24}\right)x^n = -2^{\frac{5\lambda-3}{2}}\{(1-x)(1-x^2)(1-x^3)\cdots\}^{11} + 13J,$$

and if λ be any positive even integer then

(14.4) $$\sum_0^\infty p\left(13^\lambda n + \frac{23\cdot 13^\lambda+1}{24}\right)x^n = -2^{\frac{5\lambda-7}{2}}\{(1-x)(1-x^3)(1-x^5)\cdots\}^{23} + 13J.$$

I shall reserve the discussion of these results to another paper.

A number of results such as the (??)
following can be deduced from
(14.3) and (14.4). If λ be any positive
odd integer then

$$(14.5)\quad \begin{cases} p\left(\frac{11.13^\lambda+1}{24}\right)+2^{\frac{5\lambda-3}{2}}, \\ p\left(\frac{35.13^\lambda+1}{24}\right)+2^{\frac{5\lambda-1}{2}}, \\ p\left(\frac{59.13^\lambda+1}{24}\right)-2^{\frac{5\lambda+3}{2}}, \\ p\left(\frac{83.13^\lambda+1}{24}\right)-2^{\frac{5}{2}(\lambda+1)}, \\ p\left(\frac{107.13^\lambda+1}{24}\right)-2^{\frac{5\lambda+7}{2}}, \\ p\left(\frac{131.13^\lambda+1}{24}\right)-2^{\frac{5}{2}(\lambda+1)}, \\ p\left(\frac{155.13^\lambda+1}{24}\right), \end{cases}$$

are all divisible by 13; and so on
and if λ be any positive even integer

$$(14.6)\quad \begin{cases} p\left(\frac{23.13^\lambda+1}{24}\right)+2^{\frac{5\lambda-2}{2}}, \\ p\left(\frac{47.13^\lambda+1}{24}\right)+2^{\frac{5\lambda+6}{2}}, \\ p\left(\frac{71.13^\lambda+1}{24}\right)-2^{\frac{5\lambda+2}{2}}, \\ p\left(\frac{95.13^\lambda+1}{24}\right)-2^{\frac{5\lambda+2}{2}}, \\ p\left(\frac{119.13^\lambda+1}{24}\right)-2^{\frac{5\lambda-2}{2}}, \\ p\left(\frac{143.13^\lambda+1}{24}\right)+2^{\frac{5\lambda+2}{2}}, \\ p\left(\frac{167.13^\lambda+1}{24}\right), \end{cases}$$

for modulus 13 can be completely ascertained

and so on are all divisible by 13,
In other words if n is fixed and
$\lambda+n$ is an even integer then the residue
of

$$(14.7)\quad p\left\{\frac{13^\lambda(12n-1)+1}{24}\right\}$$

(23)

General Theory

Modulus ϖ

where ϖ is a prime greater than 3.

15. We start with the two series

$$(15{\cdot}1)\qquad V_{\varpi-1}+(-1)^{\frac{\varpi-1}{2}}\,2(\varpi-1)!\,\delta_{\varpi-1}\left\{\frac{1^{\varpi-2}x}{1-x}+\frac{2^{\varpi-2}x^2}{1-x^2}+\cdots\right\}$$
$$=\Sigma K'_{\ell,m}\,Q^{\ell}R^{m},$$

where $K'_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$\ell+6m=\varpi-1;$$

and

$$(15{\cdot}2)\qquad V_{\varpi+1}+(-1)^{\frac{\varpi+1}{2}}\,2(\varpi+1)\,\delta_{\varpi+1}\left\{\frac{1^{\varpi}x}{1-x}+\frac{2^{\varpi}x^2}{1-x^2}+\cdots\right\}$$
$$=\Sigma K_{\ell,m}\,Q^{\ell}R^{m},$$

where $K_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$\ell+6m=\varpi+1.$$

In both the series V_s and δ_s are the numerator and the denominator of B_s in its lowest terms where $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, $B_6=\frac{1}{42}$, $B_8=\frac{1}{30}$, $B_{10}=\frac{5}{66}$, ... are the Bernoullis numbers.

Now by Staudt's Theorem

$$\delta_{\varpi-1}\equiv 0 \pmod{\varpi},$$

and also we have

$$n^{\varpi}-n\equiv 0 \pmod{\varpi}.$$

And so the left hand side in (15·1) is of the form

(24)

(15·3) $C' + \varpi J$

where C' is a constant integer whilst that in (15·2) is of the form

31/ (15·4) $k + CP + \varpi J$

where C and k are constant integers.

It appears that k can be taken as zero always. This involves the assertion that

(15·4) $6V_{\varpi+1} + (-1)^{\frac{\varpi+1}{2}} \frac{\varpi+1}{2} \delta_{\varpi+1} \equiv 0 \pmod{\varpi}$

I have not yet proved this result but in every particular case this can actually found to be true. Thus

31/ (15·5) can be replaced by (case)

(15·5) $CP + \varpi J$

Now using (15·3), (15·6) and (9·8) we can show in particular cases that

(15·6) $(Q^3 - R^3)^{\frac{\varpi^2-1}{24}} = x\frac{dJ}{dx} + (P^3 - R^2)\left[k_{\ell,m} P^\ell R^m\right] + \varpi J$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$4\ell + 6m = \varpi - 3.$

But it is obvious that

(15·7) $\left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{\frac{\varpi^2-1}{24}} =$

$$\frac{(1-x^{\varpi^2})(1-x^{2\varpi^2})(1-x^{3\varpi^2})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} + \varpi J.$$

It follows from (15·6) that (and (15·7))

(35)

(15.8) $\dfrac{(1-x^{\varpi q})(1-x^{\varpi q^2})(1-x^{\varpi q^3})\cdots}{(1-x)(1-x^q)(1-x^{q^2})\cdots} =$

$\left(x^{\frac{\varpi^2-1}{24}}\right)$

$$x\frac{dJ}{dx} + (R^3 - R^2)\sum k_{\ell,m} Q^\ell R^m + \varpi J$$

where the remark about the summation in (15.6) applies here also. From this we can always deduce in every particular case that

(15.9) $$\left\{\sum_1^\infty p\left(n\varpi + \varpi\left[\frac{\varpi}{24}\right] - \frac{\varpi^2-1}{24}\right)x^n + \left[\frac{\varpi}{24}\right]\right\}$$

$$= \frac{(R^3-R^2)^{\left[\frac{\varpi}{24}\right]}\sum k_{\ell,m} Q^\ell R^m + \varpi J}{(1-x^\varpi)(1-x^{2\varpi})(1-x^{3\varpi})\cdots},$$

where $k_{\ell,m}$ is a constant-integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13$$

and $[t]$ denotes as usual the greatest integer in t.

Even though all these results are very difficult to prove in general they can be easily proved when $\varpi = 23$.

i) Modulus 17, ~~and~~ 19 and 23

16. In these cases we can easily prove that

(16.1) $$\{p(5)x + p(23)x^2 + p(39)x^3 + p(56)x^4 + \cdots\}$$

$$\cdot(1-x^{17})(1-x^{34})(1-x^{51})\cdots$$

$$= 7\sum_1^\infty \tau_Q(n)x^n + 17J,$$

~~I have stated in my previous paper that~~

where $\sum_1^\infty \tau_\ell(n)x^n = Q\,x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$;

(26)

and p assumes all prime values

$$(16.2)\quad \frac{\{p(1)x + p(2^3)x^2 + p(3^3)x^3 + p(4^3)x^4 + \cdots\}}{(1-x^{15})(1-x^{24})(1-x^{57})\cdots} = 5^{-}\sum_1^\infty \tau_3(n)x^n + 19J,$$

where $\sum_1^\infty \tau_3(n)x^n = Rx\{(1-x)(1-x^3)(1-x^5)\cdots\}^{24}$;

and

$$(16.3)\quad \frac{\{p(1)x + p(2^4)x^2 + p(3^4)x^3 + p(4^4)x^4 + \cdots\}}{(1-x^{23})(1-x^{46})(1-x^{69})\cdots} = \sum_1^\infty \tau_{5^-}(n)x^n + 23J,$$

where $\sum_1^\infty \tau_{5^-}(n)x^n = QRx\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$.

I have stated without proof in my previous paper* that

$$(16.4)\quad \begin{aligned}
\sum_1^\infty \frac{\tau_2(n)}{n^s} &= \prod_p \frac{1}{1-\tau_2(p)p^{-s}+p^{15-2s}}\\
\sum_1^\infty \frac{\tau_3(n)}{n^s} &= \prod_p \frac{1}{1-\tau_3(p)p^{-s}+p^{17-2s}}\\
\sum_1^\infty \frac{\tau_4(n)}{n^s} &= \prod_p \frac{1}{1-\tau_4(p)p^{-s}+p^{23-2s}}\\
\sum_1^\infty \frac{\tau_5(n)}{n^s} &= \prod_p \frac{1}{1-\tau_5(p)p^{-s}+p^{21-2s}}\\
\sum_1^\infty \frac{\tau_7(n)}{n^s} &= \prod_p \frac{1}{1-\tau_7(p)p^{-s}+p^{25-2s}},
\end{aligned}$$

where $\sum_1^\infty \tau_4(n)x^n = Q^3x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$

and $\sum_1^\infty \tau_7(n)x^n = Q^3Rx\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}$.

All these seem to be capable of proof* as the case of $\tau(n)$ by the Mordell' method*.

Add 1.11 a 94 [3(27)] (27)

Now using (16.4) we can deduce from (16.1), (16.2) and (16.3) that

$$(16.5)\quad \{p(17^2-12)x + p(2\cdot 17^2-12)x^2 + p(3\cdot 17^2-12)x^3 + \dots\}(1-x)(1-x^2)(1-x^3)\dots = C_2 \sum_1^\infty \tau_2(n)x^n + 17J,$$

$$(16.6)\quad \{p(19^2-15)x + p(2\cdot 19^2-15)x^2 + p(3\cdot 19^2-15)x^3 + \dots\}(1-x)(1-x^2)(1-x^3)\dots = C_3 \sum_1^\infty \tau_3(n)x^n + 19J,$$

and

$$(16.7)\quad \{p(23^2-22)x + p(2\cdot 23^2-22)x^2 + p(3\cdot 23^2-22)x^3 + \dots\}(1-x)(1-x^2)(1-x^3)\dots = C_5 \sum_1^\infty \tau_5(n)x^n + 23J,$$

where C_2, C_3 and C_5 are constants. I have found that there are formulae quite analogous to these for moduli 13 even in these cases. I shall reserve the discussion of these to another paper; but I shall consider in the II part of this paper the analogous formulae for the smaller primes 5, 7 and 11. (as well as those for higher primes). The corresponding formulae for primes greater than 23 are not quite analogous. For instance in the cases of moduli 29 and 31 we have

$$(16.8)\quad \{p(13)x^3 + p(5^2)x^3 + p(81)x^4 + p(110)x^5 + \dots\}(1-x^{29})(1-x^{5^2})(1-x^{37})\dots = 8\sum_1^\infty \Omega_2(n)x^n + 29J,$$

(9.8)

where $\sum_1^\infty \Lambda_2(n) x^n = Q x^2 \{(1-x)(1-x^2)(1-x^3)\ldots\}^{48}$

and

(6.9) $\{p(22)x^2 + p(53)x^3 + p(84)x^4 + p(115)x^5 + \ldots\}$

$(1-x^{31})(1-x^{69})(1-x^{93})\ldots$

$= 10 \sum_1^\infty \Lambda_3(n) x^n + 31 T,$

where $\sum_1^\infty \Lambda_3(n) x^n = R x^2 \{(1-x)(1-x^2)(1-x^3)\ldots\}^{48}.$

The functions

$$\sum \frac{\Lambda_2(n)}{n^s}, \quad \sum \frac{\Lambda_3(n)}{n^s}$$

are obviously not capable of assuming product as in (6.4); but they are, as a matter of fact, the differences of two such products.

17. I have not investigated the residues of $\tau(n)$ for other moduli. The result which was stated before but the [illegible] 23 seems to be simple. It appears that (comparatively)

$$\sum_1^\infty \lambda(n) x^n = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{\ldots}$$

$$\{(1-x^{23})(1-x^{46})(1-x^{69})\ldots\}$$

so that

(17.1) $$\tau(n) - \lambda(n) \equiv 0 \pmod{23}$$

then

(17.2) $$\sum_1^\infty \frac{\lambda(n)}{n^s} = \frac{1}{1-23^{-s}} \Pi_1 \Pi_2 \Pi_3,$$

where

$$\Pi_1 = \prod_p \frac{1}{1-p^{-2s}},$$

p assuming all prime values of the form

(17.3) $$p \equiv 5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22 \pmod{23},$$

This can be written as $p^{11} \equiv -1 \pmod{23}$.

(29)

and $$\Pi_2 = \prod_p \frac{1}{1+p^{-s}+p^{-2s}}$$

p assuming all prime values of the form

(17.4) $p \equiv 1, 2, 3, 4, 8, 9, 12, 13, 16, 18 \pmod{23}$ *

except of the form $23a^2+b^2$ and

* This can be written as $p^{11} \equiv 1 \pmod{23}$

$$\Pi_3 = \prod_p \frac{1}{(1-p^{-s})^2}$$

p assuming all primes of the form $23a^2+b^2$. Thus $\lambda(n)$ can be completely determined and consequently the residues of $\tau(n)$ for modulus 23 can be completely ascertained.

Suppose now that

(17.5) $$\begin{cases} t_n = 0, & \tau(n) \equiv 0 \pmod{23}; \\ t_n = 1, & \tau(n) \not\equiv 0 \pmod{23} \end{cases}$$

Then it is easy to see from ( ) that

(17.6) $$\sum_1^\infty \frac{t_n}{n^s} = \Pi_1 \Pi_2 \Pi_3$$

where $$\Pi_1 = \prod_p \frac{1}{1-p^{-s}}$$

p assuming all primes of the form $23k$,

$$\Pi_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}}$$

p assuming all primes of the form (2) (17.4) except those of the form $23a^2+b^2$ and

$$\Pi_3 = \prod_p \frac{1-p^{-23s}}{(1-p^{-s})(1-p^{-23s})}$$

p assuming all primes of the form $23a^2+b^2$. (from (4)) (17.6)

It is easy to prove by quite elem

Add to a 94 (30) (30)

elementary method that

(17.7) $\tau_1 + \tau_2 + \tau_3 + \cdots + \tau_n = \underline{o}(n)$;

and by transcendental methods that

(17.8) $\tau_1 + \tau_2 + \tau_3 + \cdots + \tau_n = C\int_1^x \frac{dx}{(\log x)^{\frac{1}{2}}} + O\left(\frac{x}{(\log x)^N}\right)$

where N is any positive number and

$$C = \frac{66^{\frac{1}{2}}}{23^{\frac{3}{2}}} \cdot \frac{1-2^{-2}}{1-2^{-3}} \cdot \frac{1-3^{-2}}{1-3^{-3}} \cdot \frac{1-13^{-2}}{1-13^{-3}} \cdot \frac{1-29^{-2}}{1-29^{-3}} \cdot \cdots$$

$$\times \frac{1}{\left\{(1-5^{-2})(1-7^{-2})(1-11^{-2})(1-17^{-2})\cdots\right\}^{\frac{1}{2}}}$$

$$\times \frac{1-59^{-22}}{1-59^{-23}} \cdot \frac{1-101^{-22}}{1-101^{-23}} \cdot \frac{1-167^{-22}}{1-167^{-23}} \cdot \cdots,$$

2, 3, 13 ... being primes of the form (17.4) except those of the form $23a^2+b^2$, and 5, 7, 11, 17 ... being primes of the form (17.3) and 59, 101, 167 ... are those of the form $23a^2+b^2$. Thus we see that $\tau(n)$ is almost always divisible by 23.

We have also shown that among the values of $\tau(n)$, multiples of 3, 7 and 23 are more or less equally numerous while the multiples of 5 are less numerous than these and multiples of 2 are the most numerous.

Since

$$(1-p^{-s})(1-p^{11-s}) = (1-p^{-2s}) - (p^{11}+1)(p^{-s}-p^{-2s})$$

$$= (1-p^{-s})^2 - (p^{11}-1)(p^{-s}-p^{-2s})$$

it is easy to see from (17.2) and (17.7) that if the prime divisors of n are of the form (17.3) or of the form $23a^2+b^2$ *:

* Some may be of one form and some may be of the other form.

then

(17.9) $$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{15\,893},$$

15 893 being 23.691. If, in addition to the restrictions on the values of n in (17.9), we impose the restriction that n is odd also then it follows from ( ) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{4\,068\,608},$$

4 068 608 being 23.256.691..

Add to a 9.2 (3) (31)

Modulus 121

18. The case of modulus 11^2 seems to be much more complicated than the case of modulus 11 even though the method is theoretically the same as may be seen from the case of modulus 49. I shall now consider the case of modulus 121.

It is easy to show by using (9.2) that

$$(18.1)\quad (Q^3-R^2)^5 = P(Q^3-3R^2)(3P^3-PQ+4R) + 4Q^3(4P^3Q-3P^2R+2QR) - 26P^5+23P^3Q+16P^2R-22PQ^2+9QR+121J$$

From this we can deduce that

$$(18.2)\quad x^5\frac{\{(1-x^{11})(1-x^{22})(1-x^{33})\cdots\}^{11}}{(1-x)(1-x^2)(1-x^3)\cdots} = \sum_1^\infty \left[n^4\{a_1\sigma_1(n)+b_1\sigma_{11}(n)\} + n^3\{a_2\sigma_3(n)+b_2\sigma_{13}(n)\} + n^2\{a_3\sigma_5(n)+b_3\sigma_{15}(n)\} + n\{a_4\sigma_7(n)+b_4\sigma_{17}(n)\} + c_1n^2\tau_2(n)+c_2n\tau_3(n)+c_3\tau_4(n)\right]x^n + 121J$$

where the a's, b's and c's are constant integers and $\tau_2(n)$, $\tau_3(n)$ and $\tau_4(n)$ are the same as in (16.4). But it is easy to show that

$$(18.3)\quad \left.\begin{matrix}\tau_2(n)-n\sigma_3(n),\\ \tau_3(n)-n\sigma_5(n),\\ \tau_4(n)-n\sigma_7(n),\end{matrix}\right\} \equiv 0 \pmod{11}.$$

Add. Nu a 94[13(32)] (22) (32)

It is easy to see from (16.4) that

$$\tau_4(11n) - \tau_4(11)\tau_4(n) \equiv 0 \pmod{121}, \tag{18.4}$$

and by actual calculation we find that

$$\tau_4(11) \equiv 0 \pmod{11} \tag{18.5}$$

It is also obvious that

$$\sigma_{17}(n) - \sigma_7(n) \equiv 0 \pmod{11}. \tag{18.6}$$

Now remembering (18.3) — (18.6) and picking out the [illegible] $x^{11}, x^{22}, x^{33}, \dots$ in ( ) we obtain

$$\{p(6)x + p(17)x^2 + p(28)x^3 + p(39)x^4 + \dots\} \tag{18.7}$$
$$(1-x^{11})(1-x^{22})(1-x^{33})\dots$$
$$= 11\sum_{1}^{\infty} n.\sigma_7(n)x^n + 121J.$$

It follows from this that

$$p(121n - 5) \equiv 0 \pmod{121}, \tag{18.8}$$

and

$$p(11n-5) - p(11n-126) - p(11n-247) + p(11n-610) + \dots - 11n\sigma_7(n) \equiv 0 \pmod{121}. \tag{18.9}$$

19. In concluding the first part of this paper I shall consider the numbers which are the divisors of $\tau(n)$ for almost all values of n.

Suppose that $\varpi_1, \varpi_2, \varpi_3, \dots$ are an infinity of primes such that

$$\frac{1}{\varpi_1} + \frac{1}{\varpi_2} + \frac{1}{\varpi_3} + \dots \tag{19.1}$$

is a <u>divergent</u> series and also suppose that $a_2, a_3, a_5, a_7, \dots$ assume some or all of the positive integers (including Zero) but that $a_{\varpi_1}, a_{\varpi_2}, a_{\varpi_3}, \dots$ <u>never</u>

Add. MSS 9413(33) 27 (37)

assume the value unity. Then it is easy to show that the number of numbers of the form

$$2^{a_2}\,3^{a_3}\,5^{a_5}\,7^{a_7}\ldots \tag{19.2}$$

not exceeding n is of the form

$$\underline{o}(n). \tag{19.3}$$

In particular if a_ϖ never assumes the value unity for all prime values of ϖ of the form

$$\varpi \equiv c \pmod{k} \tag{19.4}$$

where c and k are any two integers which are prime to each other then the number of numbers of the form (19.2) is of the form

5/ (19.4') $$\underline{o}(n)$$

and more accurately is of the form

6/ (19.5) $$\underline{O}\,\frac{n}{(\log n)^{k-1}}$$

where k is the same as in (19.4).

Thus for example if s be an odd positive integer the number of values of n not exceeding n for which $\sigma_s(n)$ is not divisible by k where k is any positive integer is of the form

$$\underline{o}(n) \tag{19.7'}$$

and more accurately is of the form

$$\underline{O}\,\frac{n}{(\log n)^{k-1}}. \tag{19.8}$$

~~If k happens to be a prime number~~ For if n be written in the form

Add to (34)

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdot 7^{a_7}\cdots \qquad (34)$$

then we have

$$\sigma_s(n) = \prod_p \frac{p^{s(1+a_p)}-1}{p^s-1}, \quad p = 2, 3, 5, 7, 11, \ldots$$

Since s is odd $\sigma_s(n)$ is divisible by k if $a_p \equiv 1 \pmod{k}$ for all values of p of the form

$$p \equiv -1 \pmod{k}$$

and hence the results stated follow.

Thus we see that, if s is odd, $\sigma_s(n)$ is divisible by any given integer for almost all values of n.

It follows from all these and the formulae on §§ that

$$\tau(n) \equiv 0 \pmod{2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 23\cdot 691} \qquad (19{\cdot}9)$$

for almost all values of n.

It appears that $\tau(n)$ is almost always divisible by any power of 2, 3, and 5. It also appears from § that there are reasons to suppose that $\tau(n)$ is almost always divisible by 11 also. But I have no evidence at present to say anything about the other powers of 7 and other primes one way or the other.

Among the values of $\tau(n)$ multiples of 2, 3, 5, 7 and 23 are very numerous from the beginning but multiples of 691 begin at a very late stage. For instance $\tau(n)$ is divisible by 23 for 132 values of n not exceeding 200 which the first value of n for which $\tau(n)$ is divisible by 691 is 1381 and this is the only such value of n among the first 5000 values.

(1) Additions (Jan)

(For insertion)

(A) I have shown elsewhere* by very simple arguments that-

$$p(5n-1) \equiv 0 \pmod 5$$

$$p(7n-2) \equiv 0 \pmod 7.$$

In the case of $\tau(n)$ (of x^n) such simple arguments give the following results:

Modulus 2

It is easy to see that the coefficients in the expansions of

$$x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24}$$

and $x\{(1-x^8)(1-x^{16})(1-x^{24})\ldots\}^3$

are both odd or both even. But-

$$x\{(1-x^8)(1-x^{16})(1-x^{24})\ldots\}^3 = x - 3x^9 + 5x^{25} \ldots$$

It follows that $\tau(n)$ is odd or even according as n is an odd square or not. Thus we see that the number of values of n not exceeding n for which $\tau(n)$ is odd is only

$$\left[\frac{1+\sqrt{n}}{2}\right].$$

Modulus 5

It is easy to see that-

$$x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24} = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^4 \{(1-x^5)(1-x^{10})(1-x^{15})\ldots\}^4 + 5J$$

But the coefficient of x^{5n} in

$$x\{(1-x)(1-x^2)(1-x^3)\ldots\}^4$$

is a multiple of 5-*. It follows that-

*

(2)

(For instance)

$$\tau(5n) \equiv 0 \pmod 5$$

Modulus 7

This is the simplest of all cases. Here we have

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3 + 7J$$

$$= \{(1-x^7)(1-x^{14})(1-x^{21})\cdots\}^3 + 7J$$

But since

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^3 = x(1-3x+5x^3-7x^6+\cdots)$$

it is easy to see that the coefficients of x^{7n}, x^{7n-1}, x^{7n-2} and x^{7n-4} are ~~either~~ all multiples of 7. It follows that

$$\left.\begin{matrix}\tau(7n),\\ \tau(7n-1),\\ \tau(7n-2),\\ \tau(7n-4),\end{matrix}\right\} \equiv 0 \pmod 7.$$

Modulus 23

We have

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = x\{(1-x)(1-x^2)(1-x^3)\cdots\}$$

$$\{(1-x^{23})(1-x^{46})(1-x^{69})\cdots\} + 23J.$$

But $x(1-x)(1-x^2)(1-x^3)\cdots = \sum(-1)^V x^{1+\frac{1}{2}V(3V+1)}$

where the summation extends over all values of V from $-\infty$ to ∞. Now

$$1+\tfrac{1}{2}V(3V+1) = (6V+1)^2 - \frac{23V(3V+1)}{2}.$$

The residues of a square number for modulus 23 cannot be

5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22,—

It follows from this that

(3)

(For insertion)

$$\left.\begin{array}{l}\tau(23n-1),\\ \tau(23n-2),\\ \tau(23n-3),\\ \tau(23n-4),\\ \tau(23n+5),\\ \tau(23n-6),\\ \tau(23n+7),\\ \tau(23n-8),\\ \tau(23n-9),\\ \tau(23n+10),\\ \tau(23n+11),\end{array}\right\} \equiv 0 \pmod{23}$$

(B) so that $\lambda(n+1)$ is the number of partitions of n as the sum of integers which are not multiples of 25.

(C) If n is a multiple of 5, then

$$\tau(n) - 4830\,\tau\left(\frac{n}{5}\right) + 5^{11}\tau\left(\frac{n}{25}\right) = 0$$

in virtue of (7.6), $\tau(x)$ being considered as 0 if x is not an integer.

(D) so that $N(n+2)$ is the number of partitions of n as the sum of integers which are not multiples of 49.

(E) so that $N(n+5)$ is the number of partitions of n as the sum of integers which are not multiples of 121.

(F) Again it is easy to show by using (7.6) that

$$(10.6)\qquad \sum \frac{\tau(n)}{n^s} = \int \frac{1}{1+2^{1-s}+2^{1-2s}} \frac{1}{(1-2^{2-s})(1-3^{2-s})} \frac{1}{(1-5^{2-s})(1-5^{2-2s})} \frac{1}{(1+7^{2-s})(1-7^{2-2s})} \frac{1}{1-11^{-s}} \cdots + 11\,\mathfrak{J},$$

(4) [illegible] (38)

(For insertion)

where f is a Dirichlet's series of the form

$$\sum \frac{a_n}{n^s},$$

a_n being an integer.

From this we can deduce a number of results such as

(10.61) $$\tau(2^{4\lambda-1}n) \equiv 0 \pmod{11}$$

if n is an odd integer;

(10.62) $$\tau(3^{11\lambda-1}n) \equiv 0 \pmod{11}$$

if n is not a multiple of 3;

(10.63) $$\tau(5^{5\lambda-1}n) \equiv 0 \pmod{11}$$

if n is not a multiple of 5;

(10.64) $$\tau(7^{10\lambda-1}n) \equiv 0 \pmod{11}$$

if n is not a multiple of 7;

(10.7) $$\tau(11^{\lambda}n) - \tau(n) \equiv 0 \pmod{11}$$

and so on. Even though (10.61) — (10.64) are very analogous to one another the equations are not necessarily quite similar to these; sometimes there are more than one equation and sometimes there are equations of the form

(10.8) $$\tau(19n) \equiv 0 \pmod{11}$$

if n is not a multiple of 19, and

(10.9) $$\tau(29n) \equiv 0 \pmod{11}$$

if n is not a multiple of 29.

It is very likely that the ~~infinity of~~ primes 19, 29, ... ~~which give~~ occurring in equations like (10.8) and (10.9) are such that the sum of their reciprocals is a divergent series. If this assertion is true then $\tau(n)$ is divi-

(5) Add this as a §13 (39)

(For insertion)

sible by 11 for almost all values of n which is easily seen from §

(G₁) It follows from (11·5) that if

$$\sum_0^\infty \lambda(n)\, x^n = \frac{(1-x^9)(1-x^{18})(1-x^{27})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots},$$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 9, then

$$\lambda(n) - \sigma_1(3n+1) \equiv 0 \pmod 3.$$

Again the left-hand side of (11·4) is of the form

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} + 3J$$

while the right-hand side of (11·41) is of the form

$$\frac{1^3 x}{(1-x)^2} + \frac{2^3 x^2}{(1-x^2)^2} + \frac{3^3 x^3}{(1-x^3)^2} + \cdots + 3J.$$

It follows that

$$\tau(n) - n\,\sigma_1(n) \equiv 0 \pmod 3.$$

Suppose now that

$$\begin{cases} t_n = 0, & \lambda(n) \equiv 0 \pmod 3, \\ t_n = 1, & \lambda(n) \not\equiv 0 \pmod 3, \end{cases}$$

and that

$$\begin{cases} T_n = 0, & \tau(n) \equiv 0 \pmod 3, \\ T_n = 1, & \tau(n) \not\equiv 0 \pmod 3. \end{cases}$$

Then we can easily deduce from ( ) and ( ) that

$$\sum_0^\infty \frac{t_n}{(3n+1)^s} = \sum_1^\infty \frac{T_n}{n^s} = \Pi_1\,\Pi_2$$

where $\Pi_1 = \prod_p{}' \frac{1}{1-p^{-3s}}$

p assuming prime values of the form $3k-1$

and $\Pi_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}}$

p assuming prime values of the form $3k+1$.

We easily deduce from this that

(6) Add to a 94 (13)(23) (47)

(From inscription)

$$\begin{cases} t_1 + t_2 + t_3 + \cdots + t_n = \underline{0}(n), \\ T_1 + T_2 + T_3 + \cdots + T_n = \underline{0}(n). \end{cases}$$

In other words $\lambda(n)$ and $\tau(n)$ are divisible by 3 for almost all values of n. We can show by transcendental methods that

$$\begin{cases} t_1 + t_2 + \cdots + t_n = C\int_1^n \frac{dx}{(\log x)^{\frac{1}{2}}} + \underline{O}\frac{n}{(\log n)^r} \\ T_1 + T_2 + \cdots + T_n = \frac{C}{3}\int_1^n \frac{dx}{(\log x)^{\frac{1}{2}}} + \underline{O}\frac{n}{(\log n)^r} \end{cases}$$

where r is any positive number and

$$C = \frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}} \cdot \frac{1-7^{-2}}{1-7^{-3}} \cdot \frac{1-13^{-2}}{1-13^{-3}} \cdot \frac{1-19^{-2}}{1-19^{-3}} \cdots \times \frac{1}{\{(1-2^{-2})(1-5^{-2})(1-11^{-2})\cdots\}^{\frac{1}{2}}}$$

in both the cases, 2, 5, 7, 11 ... being primes of the form $3k-1$ and 7, 13, 19 ... being primes of the form $3k+1$.

(4) If $\sum_0^\infty \lambda(n)x^n = \frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 4 then $\lambda(n)$ is odd or even according as n is a triangular number or not.

(5) It is easy to deduce from this that $\tau(n)$ is divisible by 691 for almost all values of n, and by transcendental methods that the number of values of n not exceeding n for which $\tau(n)$ is not divisible by 691 is of the form

$$C\int_1^n \frac{dx}{(\log x)^{\frac{1}{690}}} + \underline{O}\frac{n}{(\log n)^r}$$

where C is a constant and r is any positive number.

(6) It is easy to prove that

$$x\{(1+x)(1-x^3)(1+x^5)(1-x^7)\cdots\}^{24} =$$

(7) [illegible]

(For insertion)

$$x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} + 48x^3\{(1-x^2)(1-x^4)(1-x^6)\dots\}^{24} + 2^{12}x^4\{(1-x^4)(1-x^8)(1-x^{12})\dots\}^{24}.$$

From this it is easy to deduce that

$$\tau(2n) + 24\tau(n) + 2^{11}\tau(\tfrac{1}{2}n) = 0$$

where n is any integer and $\tau(n) = 0$ if n is not an integer. Again it is easy to prove that

$$x(1+x+x^3+x^6+\dots)^8(1-2x+2x^4-2x^9+\dots)^{16} = x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24}$$

But

$$(1-2x+2x^4-\dots)^{16} = -4(1-2x+8x^4-\dots)^4 + 16(1-2x+2x^4-\dots)^2 - 11 + 256J.$$

Hence

$$x\{(1-x)(1-x^2)(1-x^3)\dots\}^{24} = 4\{1-(1-2x+2x^4-\dots)^4\}x(1+x+x^3+\dots)^8 - 16\{1-(1-2x+2x^4-\dots)^2\}x(1+x+x^3+\dots)^8 + x(1+x+x^3+x^6+\dots)^8 + 256J$$

$$= 4\{1-(1-2x+2x^4-\dots)^4\}x(1+x^2+x^6+\dots)^4 - 16\{1-(1-2x+2x^4-\dots)^2\}x(1+x^2+x^6+\dots)^4 + x(1+x+x^3+x^6+\dots)^8 + 256J$$

But

$$x(1+x+x^3+x^6+\dots)^8 = \frac{1^3x}{1-x^2} + \frac{3^3x^3}{1-x^4} + \frac{3^3x^5}{1-x^6} + \dots,$$

$$x(1+x^2+x^6+\dots)^4 = \frac{x}{1-x^2} + \frac{3x^3}{1-x^6} + \frac{5x^5}{1-x^{10}} + \dots,$$

$$x(1+x^2+x^6+\dots)^4(1-2x+2x^4-\dots)^4 = \frac{1^3x}{1-x^2} - \frac{3^3x^3}{1-x^4} + \frac{3^3x^5}{1-x^6} + \dots,$$

$$x(1+x^2+x^6+\dots)^4(1-2x+2x^4-\dots)^2 = \frac{1^2x}{1+x^2} - \frac{3^2x^3}{1+x^4} + \frac{3^2x^5}{1+x^6} - \dots$$

$$= \frac{1^2x}{1-x^2} - \frac{3^2x^3}{1-x^6} + \frac{5^2x^5}{1-x^{10}} - \dots$$

$$- \frac{3^2x^3}{1+x^4} + \frac{4^2x^4}{1+x^8} - \dots + 16J.$$

(8) Add this on 94[3(42)]

(From inversion)

It follows from all these that

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = -3\left\{\frac{1^3x}{1-x^2}+\frac{3^3x^3}{1-x^6}+\cdots\right\}$$
$$+5\left\{\frac{2^3x^2}{1-x^4}+\frac{4^3x^4}{1-x^8}+\cdots\right\}-12\left(\frac{x}{1-x^2}+\frac{2x^3}{1-x^6}+\cdots\right)$$
$$+16\left(\frac{1^2x}{1-x^2}-\frac{3^2x^3}{1-x^6}+\frac{5^2x^5}{1-x^{10}}-\cdots\right)$$
$$-16\left(\frac{2^2x^2}{1+x^4}+\frac{4^2x^4}{1+x^8}+\cdots\right)+256J.$$

Now equating only the odd powers of x we obtain

$$\tau(1)x+\tau(3)x^3+\tau(5)x^5+\cdots$$
$$=-3\left(\frac{1^3x}{1-x^2}+\frac{3^3x^3}{1-x^6}+\frac{5^3x^5}{1-x^{10}}+\cdots\right)$$
$$+16\left(\frac{1^2x}{1-x^2}-\frac{3^2x^3}{1-x^6}+\frac{5^2x^5}{1-x^{10}}-\cdots\right)$$
$$-12\left(\frac{x}{1-x^2}+\frac{3x^3}{1-x^6}+\frac{5x^5}{1-x^{10}}+\cdots\right)+256J.$$

But if n be of the form $4k+1$ then it is easy to see that

$$n''+3n^3-16n^2+12n\equiv 0 \pmod{256}.$$

Changing n to $-n$ in this formula we see that if n be of the form $4k-1$ then

$$n''+3n^3+16n^2+12n\equiv 0 \pmod{256}.$$

It follows that

$$\tau(1)x+\tau(3)x^3+\tau(5)x^5+\cdots$$
$$=\frac{1^{11}x}{1-x^2}+\frac{3^{11}x^3}{1-x^6}+\frac{5^{11}x^5}{1-x^{10}}+\cdots+256J.$$

In other words, x

$$\tau(n)-\sigma_{11}(n)\equiv 0 \pmod{256}$$

for all odd values of n, while the formula ( ) combined with this enables us to find the residues of $\tau(n)$ for modulus 2[11] for even values of n.

It follows from ( ) and ( ) that

Thus $\tau(n)+24\sigma_{11}(n)\equiv 0 \pmod{2048}$ for all values of n

(9)

(From inventions)

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{176896}$$

for all odd values of n.

(K) It follows from (13.5) that if

$$\sum_1^\infty \lambda(n)\, x^n = x^7 \cdot \frac{(1-x^{169})(1-x^{338})(1-x^{507})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots},$$

so that $\lambda(n+7)$ is the number of partitions of n as the sum of integers which are none multiples of 169, then

$$\lambda(n) - n^5\sigma(n) + 4n^4\sigma_3(n) + 3n^3\sigma_5(n) + 6n^2\sigma_7(n) + 3n\sigma_9(n) - 3\tau(n) \equiv 0 \pmod{13}.$$

The results analogous to (10.61) — (10.9) in the case of modulus 13 are

$$\tau\left(5^{13\lambda-1}n\right) \equiv 0 \pmod{13}$$

if n is not a multiple of 5;

$$\tau(7n) \equiv 0 \pmod{13}$$

if n is not a multiple of 7;

$$\tau(11n) \equiv 0 \pmod{13}$$

if n is not a multiple of 11;

$$\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}$$

if n is any integer;

$$\tau\left(19^{4\lambda-1}n\right) \equiv 0 \pmod{13}$$

if n is not a multiple of 19;

$$\tau\left(23^{3\lambda-1}n\right) \equiv 0 \pmod{13}$$

if n is not a multiple of 23;

$$\tau\left(29^{6\lambda-1}n\right) \equiv 0 \pmod{13}$$

if n is not a multiple of 29; and so on

(K) ~~It follows again from this and (~~
~~that if the n is an odd number~~
~~not that~~

$$\tau_2(n) - \sigma_{15}(n) \equiv 0 \pmod{3617}$$

$$\tau_2(n) - n\sigma_{13}(n) \equiv 0 \pmod{16170}$$

$$\tau_2(n) - 2n\sigma_9(n) + n^2\sigma_3(n) \equiv 0 \pmod{600}$$

$$\tau_3(n) - \sigma_{17}(n) \equiv 0 \pmod{43867}$$

$$\tau_3(n) - n\sigma_{15}(n) \equiv 0 \pmod{6006}$$

$$\tau_3(n) - n^2\sigma_1(n) \equiv 0 \pmod{540}$$

$$\tau_3(n) - 6n^2\sigma_9(n) + 5n\sigma_3(n) \equiv 0 \pmod{150}$$

$$\tau_3(n) + n\sigma_9(n) + n\sigma_3(n) - 5\tau(n) \equiv 0 \pmod{588}$$

$$\tau_4(n) - \sigma_{19}(n) \equiv 0 \pmod{174611}$$

Moduli $a+\frac{1}{a}$ and $a^2+\frac{1}{a^2}$

$$F(x) = \sum_0^\infty a_n x^n = \frac{(1-x)(1-x^2)\cdot(1-x^3)(1-x^4)\cdots}{(1-ax)(1-\frac{x}{a})(1-ax^2)(1-\frac{x^2}{a})\cdots}$$

$$F(x^{\frac{1}{2}}) \equiv \frac{1-x^3-x^5+\cdots}{(1+x^2)(1+x^4)\cdots} + (a-1+\tfrac{1}{a})x^{\frac{1}{2}}\frac{1-x-x^7+\cdots}{(1+x^2)(1+x^4)\cdots} \pmod{a^2+\tfrac{1}{a^2}}.$$

$\lambda_n \equiv 0 \pmod{a^2+\frac{1}{a^2}}$ when

$n =$ 2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28,
30, 31, 34, 40, 42, 45, 46, 47, 50, 53, 57,
47 values 58, 59, 62, 66, 70, 74, 75, 78, 79, 86, 94, 98,
106, 110, 122, 126, 130, 142, 154, 158, 170, 174, 206.

$\lambda_n \equiv 1 \pmod{a^2+\frac{1}{a^2}}$ when

$n =$ 14, 16, 18, 24, 32, 48, 56, 72, 82, 88, 90, 104, 114,
27 values 138, 146, 162, 178, 186, 194, 202, 210, 218, 226, 234,
242, 250, 266.

$\lambda_n \equiv -1 \pmod{a^2+\frac{1}{a^2}}$ when

$n =$ 4, 6, 12, 20, 36, 38, 44, 52, 54, 60, 68, 76, 92,
26 values 102, 118, 134, 150, 166, 182, 190, 222, 238, 254,
270, 286, 302.

$\lambda_n \equiv a-1+\frac{1}{a} \pmod{a^2+\frac{1}{a^2}}$ when

$n =$ 1, 7, 17, 23, 33, 39, 41, 49, 63, 71, 73, 81, 87,
22 values 89, 95, 105, 111, 119, 121, 127, 143, 159.

$\lambda_n \equiv -(a-1+\frac{1}{a}) \pmod{a^2+\frac{1}{a^2}}$ when

$n =$ 3, 5, 13, 29, 35, 37, 43, 51, 53, 61, 67, 69, 77, 83,
23 values 85, 91, 93, 99, 107, 115, 123, 139, 155.

$\lambda_n \equiv 0 \pmod{a+\frac{1}{a}}$ when $n =$ 11, 15, 21.

Moduli $a-1+\frac{1}{a}$ and $a+1+\frac{1}{a}$

$\lambda_n \equiv 0 \pmod{a-1+\frac{1}{a}}$ when

$n =$ 1, 6, 8, 13, 14, 17, 19, 22, 23, 25, 33, 34, 37,
19 values 44, 46, 55, 58, 61, 82.

$\lambda_n \equiv 1 \pmod{a-1+\frac{1}{a}}$ when

$n =$ 5, 7, 10, 11, 12, 18, 24, 29, 30, 31, 35, 41, 42, 43, 47, 49, 53, 54,
26 values 59, 67, 71, 73, 85, 91, 97, 109.

$\lambda_n \equiv -1 \pmod{a-1+\frac{1}{a}}$ when

$n =$ 2, 3, 4, 9, 15, 16, 20, 21, 26, 27, 28, 32, 38, 39, 40, 52, 56,
26 values 62, 64, 68, 70, 76, 94, 106, 118, 130.

$\lambda_n \equiv 0 \mod (a+1+\frac{1}{a})$ when $n = 14, 17$.

Modulus $a^3+1+\frac{1}{a^3}$

$$F(q^{\frac{1}{3}}) \equiv \frac{(1-q^2-q^7+\cdots)(1-q^4-q^5+\cdots)}{(1-q^3)(1-q^{12})(1-q^{21})\cdots}$$

$$+(a-1+\tfrac{1}{a})\,q^{\frac{1}{3}}\cdot\frac{(1-q-q^8+\cdots)(1-q^4-q^5+\cdots)}{(1-q^3)(1-q^{12})(1-q^{21})\cdots}$$

$$+(a^2+\tfrac{1}{a^2})\,q^{\frac{2}{3}}\frac{(1-q-q^8+\cdots)(1-q^2-q^7+\cdots)}{(1-q^3)(1-q^{12})(1-q^{21})\cdots}$$

$$\mod (a^3+1+\tfrac{1}{a^3})$$

$\rho_0 = a^2 - a + 1 - \frac{1}{a} + \frac{1}{a^2}$; $\rho_1 = a^2 - a + 1 - \frac{1}{a} + \frac{1}{a^2}$; $\rho_2 = a^2 + \frac{1}{a^2}$;

$\rho_3 = a^3 + 1 + \frac{1}{a^3}$; $\rho_4 = (a + \frac{1}{a})(a^2 + \frac{1}{a^2})$;

$\rho_6 = a^4 + a^2 + 1 + \frac{1}{a^2} + \frac{1}{a^4}$;

$\rho_7 = a^3 + a^2 + a + 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$

$\rho_9 = (a^2 + 1 + \frac{1}{a^2})(a^3 + 1 + \frac{1}{a^3})$

$\rho_{11} = a^6 + a^5 + a^4 + a^3 + a^2 + a + 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \frac{1}{a^6}$

Now $\sum_0^\infty \rho_n q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$ Obtain

$\rho_1 = 1$; $\lambda_1 = \rho_1$

$\rho_2 = 2$; $\lambda_2 = \rho_2$

$\rho_3 = 3$; $\lambda_3 = \rho_3$.

$\rho_4 = 5$; $\lambda_4 = \rho_5$ - $(a^4 - 2a^3 + 2a^2 - a + 1 - \frac{1}{a} + \frac{2}{a^2} - \frac{2}{a^3} + \frac{2}{a^4} - \frac{1}{a^5} + \frac{1}{a^6})$

$\rho_5 = 7$; $\lambda_5 = \rho_7 \rho$

$\rho_6 = 11$; $\lambda_6 = \rho_1 \rho_{11}$ $(a^{10} + 2a^9 + 2a^8 + 2a^7 + 3a^6 + 4a^5$

$\rho_7 = 15$; $\lambda_7 = \rho_3 \rho_5$ - $+ 6a^4 + 8a^3 + 9a^2 + 9a + 9 + \frac{9}{a} + \frac{9}{a^2}$

$\rho_8 = 22$; $\lambda_8 = \rho_1 \rho_2 \rho_{11}$ $+ \frac{8}{a^3} + \frac{6}{a^4} + \frac{4}{a^5} + \frac{3}{a^6} + \frac{2}{a^7} + \frac{2}{a^8}$

$\rho_9 = 30$; $\lambda_9 = \rho_2 \rho_3 \rho_5$ - $+ \frac{2}{a^9} + \frac{1}{a^{10}})$.

$\rho_{10} = 42$; $\lambda_{10} = \rho \rho_2 \rho_3 \rho_7$ $\rho_{14} = 135$; $\lambda_{14} = \rho_5 \rho_9$ ×

$(a^5 - a^3 + a + 1 + \frac{1}{a} - \frac{1}{a^3} - \frac{1}{a^5})$

$\rho_{15} = 176$; $\lambda_{15} = \rho_4 \rho_{11}$ $(a^7 - a^6 + a^4 + a^2 + a + \frac{1}{a}$

$+ \frac{1}{a^4} - \frac{1}{a^6} + \frac{1}{a^7})$

$\rho_{16} = 281$; $\lambda_{16} = \rho_2 \rho_7 \rho_{11}$ ×

$(a^5 - 2a^4 + 2a^3 - 2a^2 + 3a - 3 + \frac{3}{a} - \frac{3}{a^2} + \frac{3}{a^3} - \frac{3}{a^4} + \frac{1}{a^5})$.

$\rho_{17} = 297$; $\lambda_{17} = \rho_9 \rho_{11}$ ×

$(a^7 - a^6 + a^5 + a - 1 + \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^6} + \frac{1}{a^7})$

$\rho_{18} = 385$; $\lambda_{18} = \rho_5 \rho_7 \rho_{11}$ ×

$(a^6 - 2a^5 + a^4 + a^3 - a^2 + 1 - \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^4} - \frac{2}{a^5} + \frac{1}{a^6})$

$\rho_{19} = 490$; $\lambda_{19} = \rho_1 \rho_2 \rho_5 \rho_7$ ×

$(a^9 - a^7 + a^6 + 2a^5 + a^4 - 1 + \frac{1}{a^2} + \frac{2}{a^3} + \frac{1}{a^4} - \frac{1}{a^5} + \frac{1}{a^7})$

$\rho_{20} = 627$; $\lambda_{20} = \rho \rho_3 \rho_{11}$ × $(a^{10} + a^6 + a^4$

$+ a^3 + 2a^2 + 2a + 3 + \frac{2}{a} + \frac{2}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^{10}})$

$\rho_{21} = 792$; $\lambda_{21} = \rho \rho_3 \rho_4 \rho_{11}$ ×

$(a^8 - a^6 + a^4 + a + 2 + \frac{1}{a} + \frac{1}{a^4} - \frac{1}{a^6} + \frac{1}{a^8})$.

(5)

$$= x(1-3x+5x^2-7x^3+\cdots)(1-3x-x^2+x^5+\cdots)$$

$$= \sum_{\mu=0}^{\infty}\sum_{\nu=-\infty}^{\infty}(-1)^{\mu+\nu}(2\mu+1)\,x^{\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(3\nu+1)+1}$$

Now $$1+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(3\nu+1)\equiv 0 \pmod 5$$

or $$(2\mu+1)^2+2(\nu+1)^2\equiv 0 \pmod 5$$

Hence $2\mu+1$ and $\nu+1$ are both divisible by 5. It follows that the coefficient considered is either 0 or a multiple of 5. In other words

$$p_{-4}(5n-1)\equiv 0 \pmod 5$$

It follows from this and (1.1) that

(2.3) $$p(5n-1)\equiv 0 \pmod 5.$$

Precisely in the same way we can show that

$$p_{-4}\left(n\varpi-\frac{\varpi+1}{6}\right)\equiv 0 \pmod\varpi$$

where ϖ is a prime of the form $6\lambda-1$ and hence that

(2.4) $$p_{5\varpi-4}\left(n\varpi-\frac{\varpi+1}{6}\right)\equiv 0 \pmod\varpi$$

For example

(2.41) $$\begin{cases} p_{-4}(5n-1)\equiv 0 \pmod 5 \\ p_{7}(11n-2)\equiv 0 \pmod{11}. \end{cases}$$

Consider now the coefficient of x^{7n} in

$$\{(1-x)(1-x^2)(1-x^3)\cdots\}^6$$

$$= x^4(1-3x+5x^3-7x^6+\cdots)^2$$

$$= \sum\sum(-1)^{\mu+\nu}(2\mu+1)(2\nu+1)\,x^{2+\frac{1}{2}\mu(\mu+1)+\frac{1}{2}\nu(\nu+1)}$$

Now

or $$2+\tfrac{1}{2}\mu(\mu+1)+\tfrac{1}{2}\nu(\nu+1)\equiv 0 \pmod 7$$

$$(2\mu+1)^2+(2\nu+1)^2\equiv 0 \pmod 7.$$

$=$ $2\mu+1$ and $2\nu+1$ are both divisible by 7. It follows that the coefficient considered is either 0 or a multiple of 49. In other words

$$p_{-6}(7n-2)\equiv 0 \pmod{49}.$$

It follows that

(2.5) $$p(7n-2)\equiv 0 \pmod 7$$

$$\sum_0^\infty p_n x^n = 1 + \frac{x}{1-x+x^2} + \frac{x^4}{(1-x+x^2)(1-x^2+x^4)} + \cdots$$

$$p_n = \frac{1}{\sqrt{3}(n-\frac{1}{24})} \sin\left(\frac{\pi n}{3} + \frac{2\pi}{9}\right) e^{\pi\sqrt{\frac{1}{6}(n-\frac{1}{24})}}$$

$$- \frac{1}{\sqrt{6}(n-\frac{1}{24})} \cdot 2\sin\left(\frac{\pi n}{3} - \frac{\pi}{9}\right)\cos\left(\frac{\pi n}{2} + \frac{\pi}{8}\right) \times e^{\frac{\pi}{2}\sqrt{\frac{1}{6}(n-\frac{1}{24})}} + \cdots$$

$p_0 = 1.$ $p_7 = 1$ $p_{18} = 1.$ $p_{19} = 1.$ $p_{25} = 1$ $p_{31} = 1$
$p_{37} = 2.$ $p_{43} = 1.$ $p_{49} = 2.$ $p_{55} = 2.$

$p_2 = 1.$ $p_8 = 0.$ $p_{14} = 1.$ $p_{20} = 0.$ $p_{26} = 1.$ $p_{32} = 0$
$p_{38} = 1.$ $p_{44} = 0.$ $p_{50} = 1.$ $p_{56} = 1.$

$p_3 = 0.$ $p_9 = 0.$ $p_{15} = -1.$ $p_{21} = 0.$ $p_{27} = 0.$ $p_{33} = -1$
$p_{39} = -1.$ $p_{45} = -1.$ $p_{51} = +1.$ $p_{57} = -1.$

$p_4 = 0.$ $p_{10} = -1.$ $p_{16} = 0.$ $p_{22} = -1.$ $p_{28} = -1.$ $p_{34} = -1$
$p_{40} = -1.$ $p_{46} = -2.$ $p_{52} = -1.$ $p_{58} = -2.$

T.O

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c.o.

$p_3 = 0,\ p_{11} = 0,\ p_{17} = 0,\ p_{23} = 0,\ p_{29} = -1,\ p_{35} = 0,$

$p_{41} = 0,\ p_{47} = 0,\ p_{53} = -1,\ p_{59} = -1,$

$p_6 = 1,\ p_{12} = 1,\ p_{18} = 0,\ p_{24} = 1,\ p_{30} = 1,\ p_{36} = 1,$

$p_{42} = 1,\ p_{48} = 1,\ p_{54} = 1,\ p_{60} = 0.$

$p_3, p_4, p_5, p_8, p_9, p_{11}, p_{16}, p_{17}, p_{18}, p_{20}, p_{21}, p_{23}, p_{27},$
$p_{32}, p_{35}, p_{41}, p_{44}, p_{47}$ are the only zeroes.

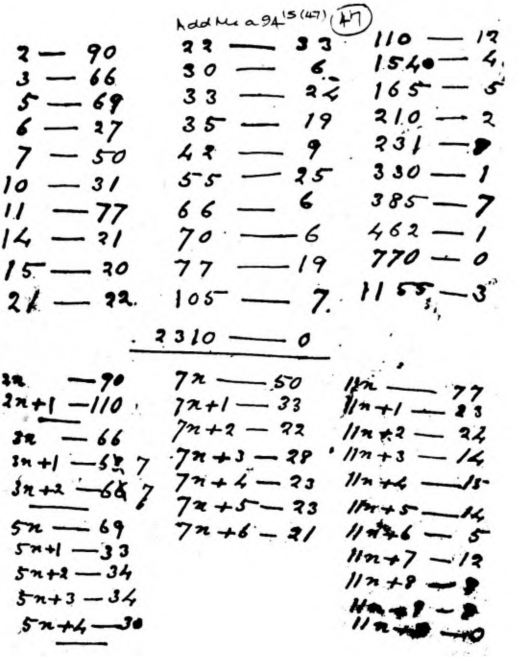
If $\sum_0^\infty \lambda_n v^n = 1 + \frac{v}{1+\sqrt{5}v+v^2} + \frac{v^4}{(1+\sqrt{5}v+v^2)(1+\sqrt{5}v^2+v^4)} + \frac{v^9}{\cdots} + \cdots \times \frac{}{(1+\sqrt{5}v^2+v^4)}$

then

$\lambda_1 + \lambda_6 v + \lambda_{11} v^2 + \lambda_{16} v^3 + \cdots$
$= \frac{(1-v^5)(1-v^{10})(1-v^{15}) \cdots}{(1-v)(1-v^4)(1-v^6) \cdots}$

$\lambda_2 + \lambda_7 v + \lambda_{12} v^2 + \lambda_{17} v^3 + \cdots$
$= -\frac{\sqrt{5}+1}{2} \cdot \frac{(1-v^5)(1-v^{10})(1-v^{15}) \cdots}{(1-v^2)(1-v^3)(1-v^7) \cdots}$

λ_{5n-1} is identically zero.



A Sheet from the 'LOST NOTEBOOK'

$11n+10$. 10000 11000, 10100 1111 0,
1111111111 1, 2122222222,
3232333343, 4444555555.

$11n+9$. 1000100110, 0010111011,
1111111121, 2122222 3222,
3233433344, 4454555656,
7677787888 98.

$11n+8$. 01.10.0010000, 0001000001,
~~0100010001~~
0100010001, 0101011101,
1101011112, 1112111211.

$11n+7$. 1001001011, 0010111111,
1.1112 1221, 2223 2233 3,
3343444454, 5555566777 7,
8789999.10.9. 10.11, 1111.

$11n+5$. 10100110 10, 1110 1111 21,
$11 q^{62}$
$11 q^{64}$ 2111222222, 3232334343,
4454555656, 7677879899.

$11n+4$. 10101011 1, 1011111121,
$11 q^{60}$
$11 q^{73}$ 2121 2132 32, 3233334344,
5454656576, 7687998 99.10.9.

$11n+3$

$n = 0, 1, 2, 3, 4, 5, \ldots$

$\|n -$ 1 1 1 1 1 1 1 1 1 1, 2 1 2 2 2 2 2 2 2 2,
3 3 3 3 4 4 4 4 4 4, 5 5 5 6 6 6 7 7 7 7,
5068 8 8 9 9. 10. 10 0 0 1, 2 2 3 3 4 5 6 6 7 7,
517 9 9. 10. 1 2 3 4 5 6, 8 8. 10. 0 1 2 4 5 7 8,
693 8
803 10 0 2 3 6 7 9 0 2 4, 7 8 0 2 5 6. 10. 1 4 6, 10.
891 957 1012

$\|n+1-$ 1 0 1 1 1 0 1 0 1 1, 1 1 2 1 1 1 2 1 2 1,
2 2 2 2 3 2 3 3 3 3, 4 3 4 3 4 4 5 4 5 5,

$\|n+2-$ 1 1 1 0 1 1 1 1 1, 1 1 2 1 2 1 2 2 2 2,
2 3 3 2 3 3 3 3 4 4, 4 4 5 4 5 5 6 6 6 6.

$\|n+3-$ 1 0 0 1 1 0 1 1 1 1, 1 1 1 1 1 2 2 1 2 2,
2 2 2 2 3 3 2 3 3 3, 4 4 3 4 4 4 5 5 5,

$\|n+4-$ 1 0 1 0 1 0 1 0 1, 1 0 1 1 1 1 1 1 2 1,
2 1 2 1 2 1 3 2 3 2, 3 2 3 3 3 3 4 3 4 4,

$\|n+5-$ 1 0 1 0 0 1 1 0 1 0, 1 1 1 0 1 1 1 1 2 1,
2 1 1 1 2 2 2 2 2 2, 3 2 3 2 3 3 4 3 4 3.

$\|n+6-$ 0 0 0 0 0 0 0 0 all zeros.

$\|n+7-$ 1 0 0 1 0 0 1 0 1 1, 0 0 1 0 1 1 1 1 1 1,
~~1 1 1 2 1 1 1 2 2 1 2~~,
1 1 1 1 2 1 1 2 2 1, 2 2 2 3 2 2 2 3 3 3,

$\|n+8-$ 0 1. 10. 0 0 1 0 0 0 0, 0 0 0 1 0 0 0 0 0 1,
0 1 0 0 0 1 0 0 0 1, 0 1 0 1 0 1 1 1 0 1,

$\|n+9-$ 1 0 0 0 1 0 0 1 1 0, 0 0 1 0 1 1 1 0 1 1,
1 1 1 1 1 1 1 1 2 1, 2 1 2 2 2 2 3 2 2 2,

$\|n+10-$ 1 0 0 0 0 1 1 0 0 0, 1 0 1 0 0 1 1 1 1 0,
1 1 1 1 1 1 1 1 1 1, 2 1 2 2 2 2 2 2 2 2,

$$\frac{1-5^{-1}x+7^{2}x^{2}+}{1-x-x^{2}+} = P$$

$$\frac{1-5^{-4}x-}{1-} = 3P^{2}-2Q$$

$$\frac{2(P^{2}-Q)+P^{2}}{1-5^{6}x-} = 15P^{3}-30PQ+16R$$

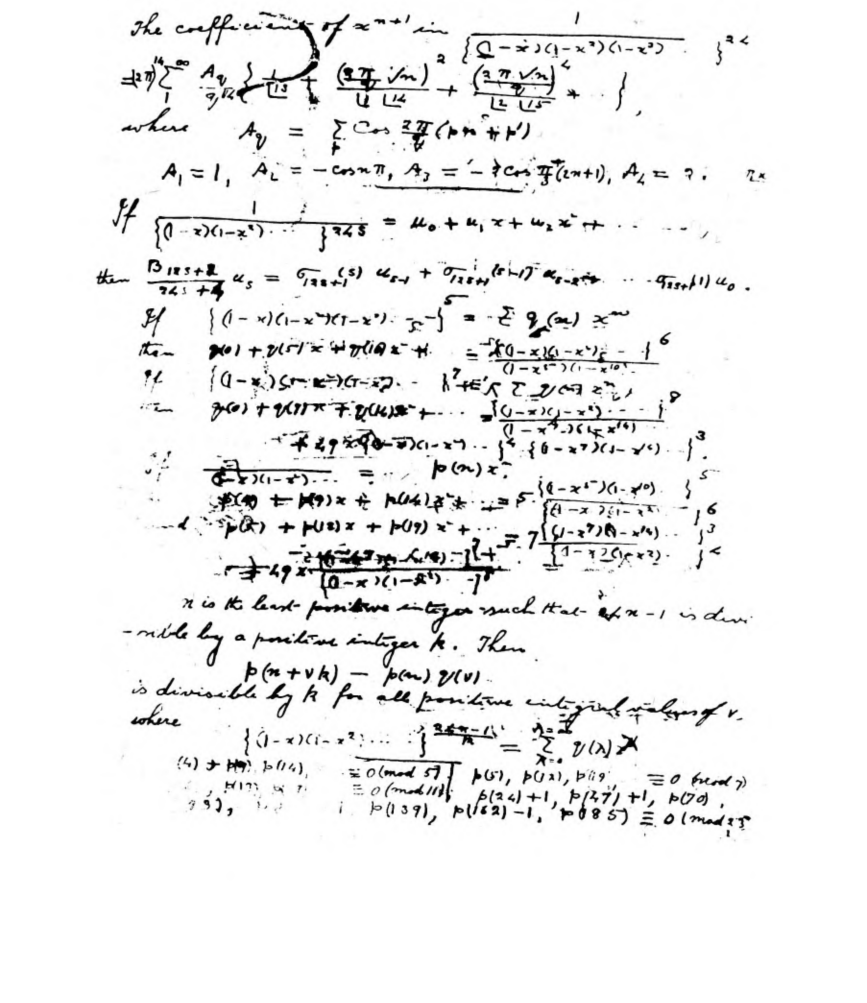
$$\frac{1-5^{8+2}x-f}{1-x-x^{2}} = 24\,\frac{xd}{dx}\cdot\frac{1-5^{3}x-}{} + P\cdot\frac{1-5^{5}x-}{1-x^{2}}$$

$$\frac{1-5^{5}x}{1-x-} = 105P^{4}-420P^{2}Q+448PR-132Q^{2}$$

$$\frac{1-5^{10}x-}{1-x-x^{2}} = 945P^{5}-6300P^{3}Q+10080P^{2}R$$
$$-5940PQ^{2}+1216QR$$

$$\frac{1-5^{12}x+}{} = 10395P^{6}-103950P^{4}Q$$
$$+221760P^{3}R-196020P^{2}Q^{2}$$
$$+82256PQR-(27720Q^{3}$$
$$+9728R^{2})$$

$$1\cdot3\cdot5\cdot7\cdots(2s-1)\left\{P^{s}-\frac{s(s-1)}{3}P^{s-2}Q\right.$$
$$+\frac{8s(s-1)(s-2)}{45}P^{s-3}R-\frac{11s(s-1)(s-2)(s-3)}{210}P^{s-4}$$
$$+\frac{15\cdot7s(s-1)(s-2)(s-3)(s-4)}{29925}P^{s-5}QR$$
$$/4175$$
$$-\phi(s)P^{s-3}\}\phi(s)Q^{2}+$$



A Sheet from the 'LOST NOTEBOOK'

4. There are of course results corresponding to all the previous results for the functions analogous to the ζ-function. Thus for example we have, 14

$$(18)\qquad \int_0^\infty \frac{\cos 2nx}{\cosh \pi x}\,dx = \frac{1}{2\cosh n}$$

$$(19)\qquad \int_0^\infty \frac{\cos 2nx}{1+2\cosh \pi x}\,dx = \frac{1}{\sqrt{3}}\cdot\frac{1}{1+2\cosh 2n}$$

$$(20)\qquad \int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x}\cos 4nx\,dx = \frac{1}{2\sqrt{2}}\cdot\frac{\cosh n}{\cosh 2n}$$

$$(21)\qquad \int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x}\sin 4nx\,dx = \frac{1}{2\sqrt{2}}\cdot\frac{\sinh n}{\cosh 2n}$$

$$(22)\qquad \int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x-1}\sin 6nx\,dx = \frac{1}{2\sqrt{3}}\cdot\frac{\sinh n}{2\cosh 2n-1}$$

From these we can easily deduce that

$$(23)\qquad \sqrt{\alpha}\int_0^\infty \frac{e^{-(\alpha x)^2}}{\cosh \pi x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-(\beta x)^2}}{\cosh \pi x}\,dx$$

with the condition that $\alpha\beta = \pi$.

(24) $$\sqrt{\alpha}\int_0^\infty \frac{e^{-(\alpha x)^2}}{2\cosh \pi x + 1}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-(\beta x)^2}}{2\cosh \pi x + 1}\,dx$$

with the condition $\alpha\beta = \frac{3}{4}\pi$.

(25) $$\sqrt{\alpha}\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} e^{-(\alpha x)^2} dx = \sqrt{\beta}\int_0^\infty \frac{\cosh \pi x}{\cosh 2\pi x} e^{-(\beta x)^2} dx$$

with the condition $\alpha\beta = 2\pi$.

(26) $$\sqrt{\alpha}\int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x}\cdot \alpha x e^{-(\alpha x)^2} dx = \sqrt{\beta}\int_0^\infty \frac{\sinh \pi x}{\cosh 2\pi x}\,\alpha x\, e^{-(\alpha x)^2} dx$$

with the condition $\alpha\beta = 2\pi$.

(27) $$\sqrt{\alpha}\int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1}\cdot \alpha x\, e^{-(\alpha x)^2} dx = \sqrt{\beta}\int_0^\infty \frac{\sinh \pi x}{2\cosh 2\pi x - 1}\cdot \beta x\, e^{-(\beta x)^2} dx$$

with the condition $\alpha\beta = 3\pi$.

From the above results we can easily deduce the results corresponding to those of §§ 2 - 3 for the functions

$$1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} - \cdots\cdots$$
$$1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + 7^{-s} - \cdots\cdots$$
$$1^{-s} + 3^{-s} - 5^{-s} - 7^{-s} + 9^{-s} + \cdots\cdots$$
$$1^{-s} - 3^{-s} - 5^{-s} + 7^{-s} + 9^{-s} - \cdots\cdots$$
$$1^{-s} - 5^{-s} - 7^{-s} + 11^{-s} + 13^{-s} - \cdots\cdots,$$

where 1, 3, 5, 7 ... are the natural odd numbers, 1, 2, 4, 5 ... are the natural numbers without the multiples of 3 and 1, 5, 7, 11 ... are the natural odd numbers without the multiples of 3.

28. $1^{-s} + 3^{-s}\omega\varpi + 5^{-s}\omega^2 + 7^{-s}\omega^2 + 11^{-s}\omega^4\varpi + 13^{-s}\omega^3 +$
$+ 15^{-s}\varpi + 17^{-s}\omega + 19^{-s}\omega^5\varpi + 23^{-s}\omega^2\varpi + 25^{-s}\omega^4 +$
$+ 27^{-s}\omega^3\varpi + \cdots$

where $\omega^6 = \varpi^2 = 1$.

32. $1^{-s} + 3^{-s}\omega + 5^{-s}\omega^3\varpi + 7^{-s}\omega^6\varpi + 9^{-s}\omega^2 + 11^{-s}\omega^7$
$+ 13^{-s}\omega^5\varpi + 15^{-s}\omega^4\varpi + 17^{-s}\omega^4 + 19^{-s}\omega^5 + 21^{-s}\omega^7\varpi$
$+ 23^{-s}\omega^2\varpi + 25^{-s}\omega^6 + 27^{-s}\omega^3 + 29^{-s}\omega\varpi + 31^{-s}\varpi + \cdots$

where $\omega^8 = \varpi^2 = 1$.

36. $1^{-s} + 5^{-s}\omega + 7^{-s}\omega^2\varpi + 11^{-s}\omega^5\varpi + 13^{-s}\omega^4 + 17^{-s}\omega^3 + 19^{-s}\varpi$
$+ 23^{-s}\omega\varpi + 25^{-s}\omega^2 + 29^{-s}\omega^5 + 31^{-s}\omega^4\varpi + 35^{-s}\omega^2\varpi + \cdots$

where $\omega^6 = \varpi^2 = 1$.

40.

$$\frac{\sqrt{(1-s)}}{\sqrt{(s)}}$$

17 Add to a 7th S/S.

$$\int_0^\infty e^{-mx}\cos nx\,dx = \int_0^\infty e^{-nx}\sin mx\,dx.$$

Hence $$\int_0^\infty\int_0^\infty e^{-mx} f(m)\cos nx\,dm\,dx = \int_0^\infty\int_0^\infty e^{-nx} f(m)\sin mx\,dm\,dx.$$

In other words, if

~~If $\int_0^\infty f(x)\sin nx\,dx = \phi(n)$, $\int_0^\infty f(x)\,e^{-nx}\,dx = \psi(n)$~~

I $$\int_0^\infty e^{-nx} f(x)\,dx = \phi(n), \quad \int_0^\infty f(x)\sin nx\,dx = \psi(n)$$

then $$\int_0^\infty \phi(x)\cos nx\,dx = \int_0^\infty e^{-nx}\psi(x)\,dx$$

$$\int_0^\infty \tfrac{1}{2}\phi(\tfrac{1}{2})\sin nx\,dx = \int_0^\infty \tfrac{1}{2}\psi(\tfrac{1}{2})e^{-nx}\,dx.$$

Again $$\int_0^\infty\int_0^\infty e^{-mx} f(n)\cos nx\,dn\,dx = \int_0^\infty\int_0^\infty e^{-nx} f(n)\sin mx\,dn\,dx$$

In other words, if

II $$\int_0^\infty e^{-nx} f(x)\,dx = \phi(n), \quad \int_0^\infty f(x)\cos nx\,dx = \psi(n)$$

then $$\int_0^\infty \phi(x)\sin nx\,dx = \int_0^\infty e^{-nx}\psi(x)\,dx.$$

$$\int_0^\infty \tfrac{1}{x}\phi(\tfrac{1}{x})\cos nx\,dx = \int_0^\infty \tfrac{1}{x}\psi(\tfrac{1}{x})\,e^{-nx}\,dx$$

If $$\int_0^\infty \phi(x)\cos nx\,dx = k\sqrt{(\tfrac{n}{2})}\,\phi(k^2 n)$$

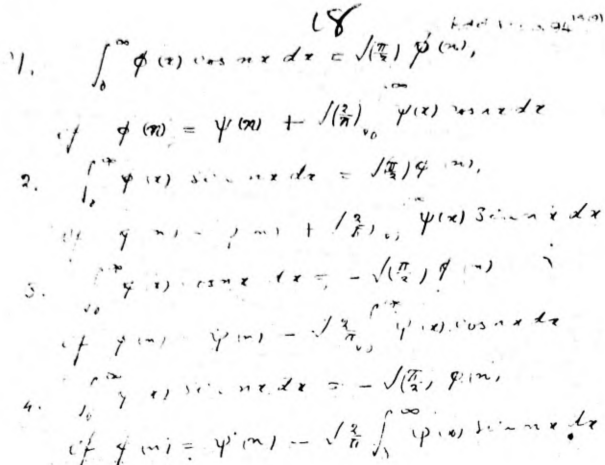
and $$\int_0^\infty e^{-nx}\phi(x)\,dx = \psi(n)$$

then $$\int_0^\infty \psi(x)\sin nx\,dx = \tfrac{1}{k}\sqrt{(\tfrac{n}{2})}\,\psi(\tfrac{n}{k^2}x)$$

If $$\int_0^\infty \phi(x)\sin nx\,dx = k\sqrt{(\tfrac{n}{2})}\,\phi(k^2 n)$$

and $$\int_0^\infty e^{-nx}\phi(x)\,dx = \psi(n)$$

then $$\int_0^\infty \psi(x)\cos nx\,dx = \tfrac{1}{k}\sqrt{(\tfrac{n}{2})}\,\psi(\tfrac{n}{k^2}).$$



A Sheet from the 'LOST NOTEBOOK'

11

If $\alpha\beta = 1$, and $\phi(x) = -\frac{1}{2x} + \frac{\Gamma'(x)}{\Gamma(x)} - \gamma - \log x$, where γ is Euler's constant, then

$$\sqrt{\alpha}\left\{\frac{\gamma}{2\alpha} - \frac{\log 2\pi\alpha}{2\alpha} + \phi(\alpha) + \phi(2\alpha) + \phi(3\alpha) + \cdots\right\}$$

$$= \sqrt{\beta}\left\{\frac{\gamma}{2\beta} - \frac{\log 2\pi\beta}{2\beta} + \phi(\beta) + \phi(2\beta) + \phi(3\beta) + \cdots\right\}$$

$$= -\sqrt{\alpha}\int_0^\infty \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)\left(e^{2\pi\alpha x} - \frac{1}{2\pi\alpha x}\right) dx$$

$$= -\frac{1}{\pi\sqrt{\pi}}\int_0^\infty \left|\Xi\left(\tfrac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt.$$

(1) $e^{-\pi x} + \frac{1}{2} e^{-4\pi x} + \frac{1}{3} e^{-9\pi x} + \cdots$

$= \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - 4\pi^2\sqrt{x} \sum_{1}^{\infty} \int_0^\infty \alpha z e^{-\pi x(n+z)^2} dz$

$R(\alpha) \geq 0.$

(2) $\cos \pi x + \frac{1}{2} \cos 4\pi x + \frac{1}{3} \cos 9\pi x + \cdots$

$= \frac{\pi}{6}^2 - \pi\sqrt{(\frac{1}{2}x)} + 4\pi^2\sqrt{x} \sum_{1}^{\infty} \int_0^\infty xz e^{-2\pi\pi z} \cos\left(\frac{\pi}{4} - \pi n^2 z + \pi x z^2\right) dx$

(3) $\sin \pi x + \frac{1}{2} \sin 4\pi x + \frac{1}{3} \sin 9\pi x + \cdots$

$= \pi\sqrt{(\frac{1}{2}x)} - \frac{1}{2}\pi x + 4\pi^2\sqrt{x} \sum_{1}^{\infty} \int_0^\infty xz e^{-2\pi\pi z} \sin\left(\frac{\pi}{4} - \pi n^2 z + \pi x z^2\right) dz.$

In (2) and (3) $x \geq 0$

If a is an odd positive integer then

(i) $\cos \frac{\pi}{a} + \frac{1}{4} \cos \frac{4\pi}{a} + \frac{1}{9} \cos \frac{9\pi}{a} + \cdots$

$= \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{n=1}^{n=a} \frac{n}{a}\left(1 - \frac{n}{a}\right) \sin\left(\frac{\pi}{4} + \pi\frac{n^2}{a}\right)$

(ii) $\sin \frac{\pi}{a} + \frac{1}{4} \sin \frac{4\pi}{a} + \frac{1}{9} \sin \frac{9\pi}{a} + \cdots$

$= -\frac{\pi^2}{\sqrt{a}} \sum_{n=1}^{n=a} \frac{n}{a}\left(1 - \frac{n}{a}\right) \cos\left(\frac{\pi}{4} + \pi\frac{n^2}{a}\right)$

(iii) $\sin\left(\frac{\pi}{4} + \frac{\pi}{a}\right) + \frac{1}{4} \sin\left(\frac{\pi}{4} + \frac{4\pi}{a}\right) + \cdots$

$= \frac{\pi^2}{6\sqrt{2}} + \frac{4\pi^2}{a\sqrt{a}} \left\{ \frac{1}{8\pi} + \frac{\cos(\pi/a)}{e^{2\pi} - 1} + \frac{2\cos 4\pi}{e^{4\pi} - 1} + \cdots \right\}$

$- 2\pi^2\sqrt{2} \left\{ \frac{1}{8\pi a} + \frac{1}{e^{2\pi a} + 1} + \frac{2}{e^{4\pi a} + 1} + \cdots \right\}$

$= \frac{\pi^2}{6\sqrt{2}} - \frac{\pi^2}{\sqrt{a}} \sum_{n=1}^{n=a} \frac{n}{a}\left(1 - \frac{n}{a}\right) \cos \frac{\pi n^2}{a} \cdots$

$e^{-\frac{\pi}{4}x} - \frac{1}{8} e^{-\frac{9\pi}{4}x} + \frac{1}{5} e^{-\frac{25\pi}{4}x} - \cdots$

$= \frac{\pi}{4} \pi\sqrt{x} \sum_{0}^{\infty} \int_0^\infty (-1)^n e^{-\pi x\left(n + \frac{1}{2} + x z\right)^2} dz$

$$(1)\int_0^\infty \frac{\sin \pi t x}{x \cosh \pi x}\, e^{\pm i\pi n x^2}\, dx$$

$$= \frac{\pi}{2} - 2\left(e^{-\frac{1}{2}t + \frac{1}{2}i\pi n} - \frac{1}{3}e^{-\frac{3}{2}t + \frac{3}{2}i\pi n} + \cdots\right)$$

$$- \frac{\pi}{\sqrt{n}}\, e^{-\frac{i\pi}{4}} \int_0^\infty \left\{ e^{\frac{i\pi}{4n}(t+z+i)^2} - e^{\frac{i\pi}{4n}(t+z+3i)^2} + \cdots \right\} dz$$

$$F_\omega(t) = \int_0^\infty \frac{\sin \pi t x}{\cosh \pi x} e^{-\pi x^2/\omega} \, dx$$

$$= \frac{e^{-\pi t^2/4\omega}}{\sqrt{\omega}} \int_0^\infty \frac{\sin \pi \frac{t}{\omega} x}{\cosh \pi x} e^{-\pi x^2/\omega} \, dx$$

$$F_\omega(t) = -F_\omega(-t), \quad F_\omega(t) = -\frac{i e^{-\pi t^2/4\omega}}{\sqrt{\omega}} F_{\frac{1}{\omega}}\left(\frac{t}{\omega}\right).$$

$$F_\omega(t) - F_\omega(t+2m\omega)$$

$$= -\frac{i}{\sqrt{\omega}} \left\{ \frac{1}{2} e^{-\frac{\pi}{4\omega} t^2} + e^{-\frac{\pi}{4\omega}(t+2\omega)^2} + e^{-\frac{\pi}{4\omega}(t+4\omega)^2} + \cdots \right.$$

$$F_\omega(t) - F_\omega(t+2m\omega)$$

$$= e^{-\frac{\pi t^2}{4\omega}} \left\{ \frac{1}{2} e^{\frac{\pi}{4\omega} t^2} + e^{\frac{\pi}{4\omega}(t+2\omega)^2} + \cdots \right.$$

If $s = t + 2\eta_1 m\omega + 2\eta_2 n i$, $\eta_1^2 + \eta_2^2 = 1$

$$F_\omega(s) + (-1)^{mn-1} e^{-\frac{1}{2}\pi \eta_1 m (s+t)} F_\omega(t)$$

$$= e^{-\frac{\pi s^2}{4\omega}} \left\{ \frac{1}{2} e^{\frac{\pi}{4\omega} s^2} + e^{\frac{\pi}{4\omega}(s-2\eta_1\omega)^2} + \cdots \right.$$

$$+ (-1)^{mn} \frac{i}{\sqrt{\omega}} e^{-\frac{1}{2}\pi \eta_1 m (s+t)} \left\{ \frac{1}{2} e^{-\frac{\pi}{4\omega} t^2} + e^{-\frac{\pi}{4\omega}(t+2\eta_2 i)^2} + \cdots \right.$$

Add to a 94 (15)

$a > 0 \quad h \geq 0.$

$$\int_{-h}^{\infty} \frac{a^x}{\Gamma(1+x)}\,dx + \int_0^{\infty} \frac{e^{-ax}\,x^{h-1}}{\pi^2 + (\log x)^2}\left(\cos \pi h - \frac{1}{\pi}\sin \pi h \log x\right) dx$$

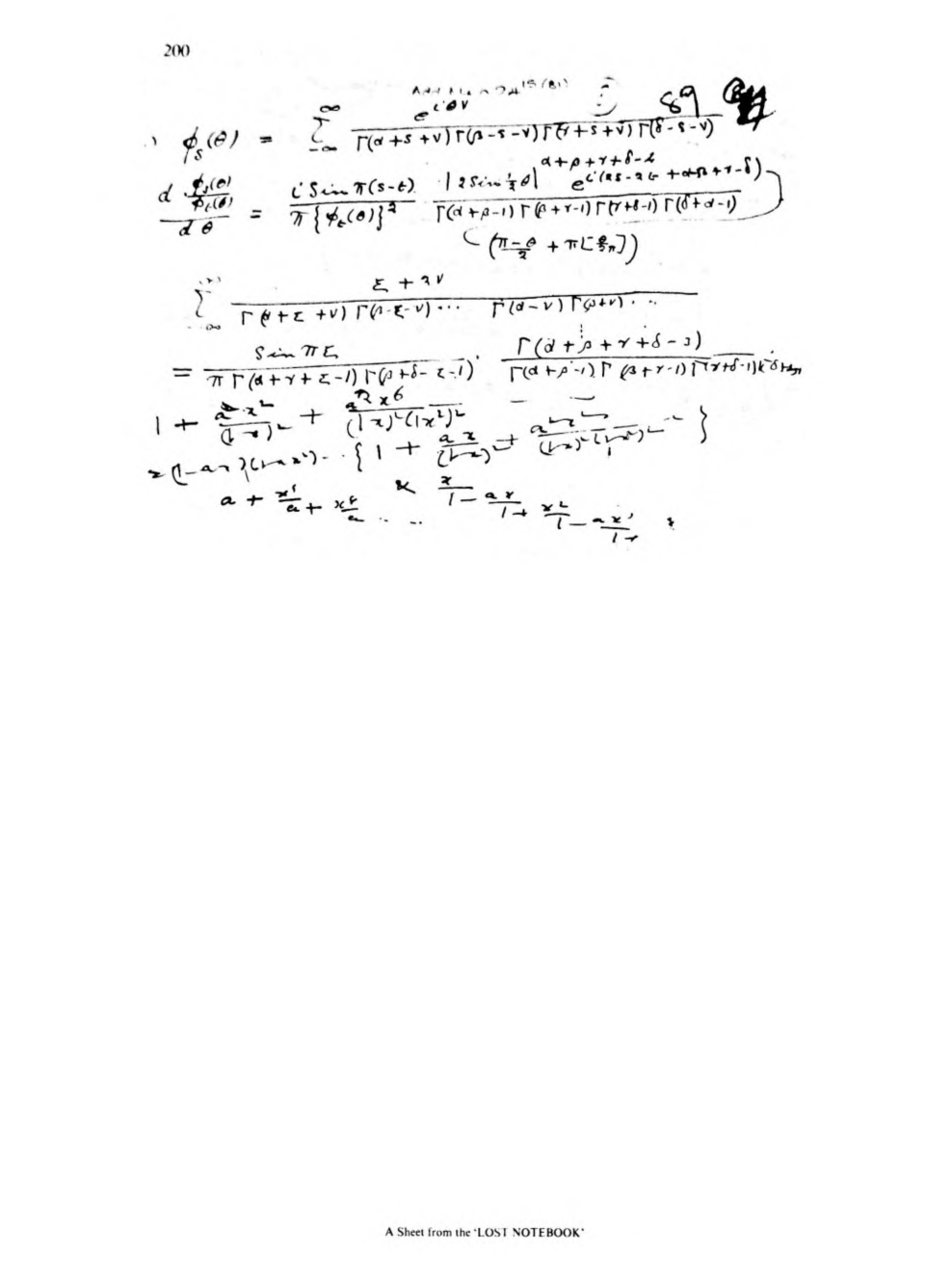
$$= e^a$$

If $a > 0$, $\lambda < \varepsilon$, where $\frac{1}{\varepsilon}$ is a positive integer

$$\varepsilon\left\{\frac{a^\lambda}{\Gamma(1+\lambda)} + \frac{a^{\lambda+\varepsilon}}{\Gamma(1+\lambda+\varepsilon)} + \frac{a^{\lambda+2\varepsilon}}{\Gamma(1+\lambda+2\varepsilon)} + \cdots\right\}$$

$$= e^a - \frac{\varepsilon}{\pi}\int_0^{\infty} e^{-ax}\,x^{-\lambda-1}\cdot\frac{\sin \pi(\lambda-\varepsilon) - x^\varepsilon \sin \pi\lambda}{2\cos \pi\varepsilon - (x^\varepsilon + x^{-\varepsilon})}\,dx$$

$$e^a = \int_{-h}^{\infty} \frac{a^x}{\Gamma(1+x)}\,dx + \frac{1}{2\pi}\int_0^{\infty} \frac{e^{i\pi(h+ix)}}{a^{h+ix}}\Gamma(h+ix) + \frac{e^{-i\pi(h-ix)}}{a^{h-ix}}\Gamma(h-ix)\,dx$$



$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(1+ae^{-h^2+2hx})(1+ae^{-h^2+2h+2hx})\cdots dx$$

$$\sqrt{\pi}\left\{e^{m^2}+ae^{(m+h)^2}\frac{\lambda}{1-\lambda}+a^2e^{(m+2h)^2}\frac{\lambda^3}{(1-\lambda)(1-\lambda^2)}\right.$$

$$\left.\frac{a^\nu e^{(m+\nu h)^2}\lambda^{\frac{\nu(\nu+1)}{2}}}{}=\zeta^\nu e^{m^2}\frac{1+e^{(m+h)^2-\beta h^2}}{a^\nu\ e^{(m+\nu h)^2-\nu\beta h^2}}\right.$$

$$\nu^2h^2=\frac{\beta^2}{2}\left\{\log\lambda+2\pi i\,\xi\right.$$

$$\begin{cases}2k^2=\log\lambda+2\pi i\,\xi\\ \log a+2mk+\frac{1}{2}\log\lambda=\log\zeta+2\pi i\,\tau\end{cases}$$

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(1+ae^{-\beta h^2+2hx}\cdot(1+ae^{-3\beta h^2+2hk})\cdots dx$$

$$=\sqrt{\pi}\,e^{m^2}\cdot\frac{1}{(1-ae^{2mh})(1-ae^{2mh-2h^2})(1-ae^{2mh-4h^2})}$$

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}\frac{dx}{(1-ae^{-2h^2+2ihx})(1-ae^{-2h^2+2ihx})\cdots}$$

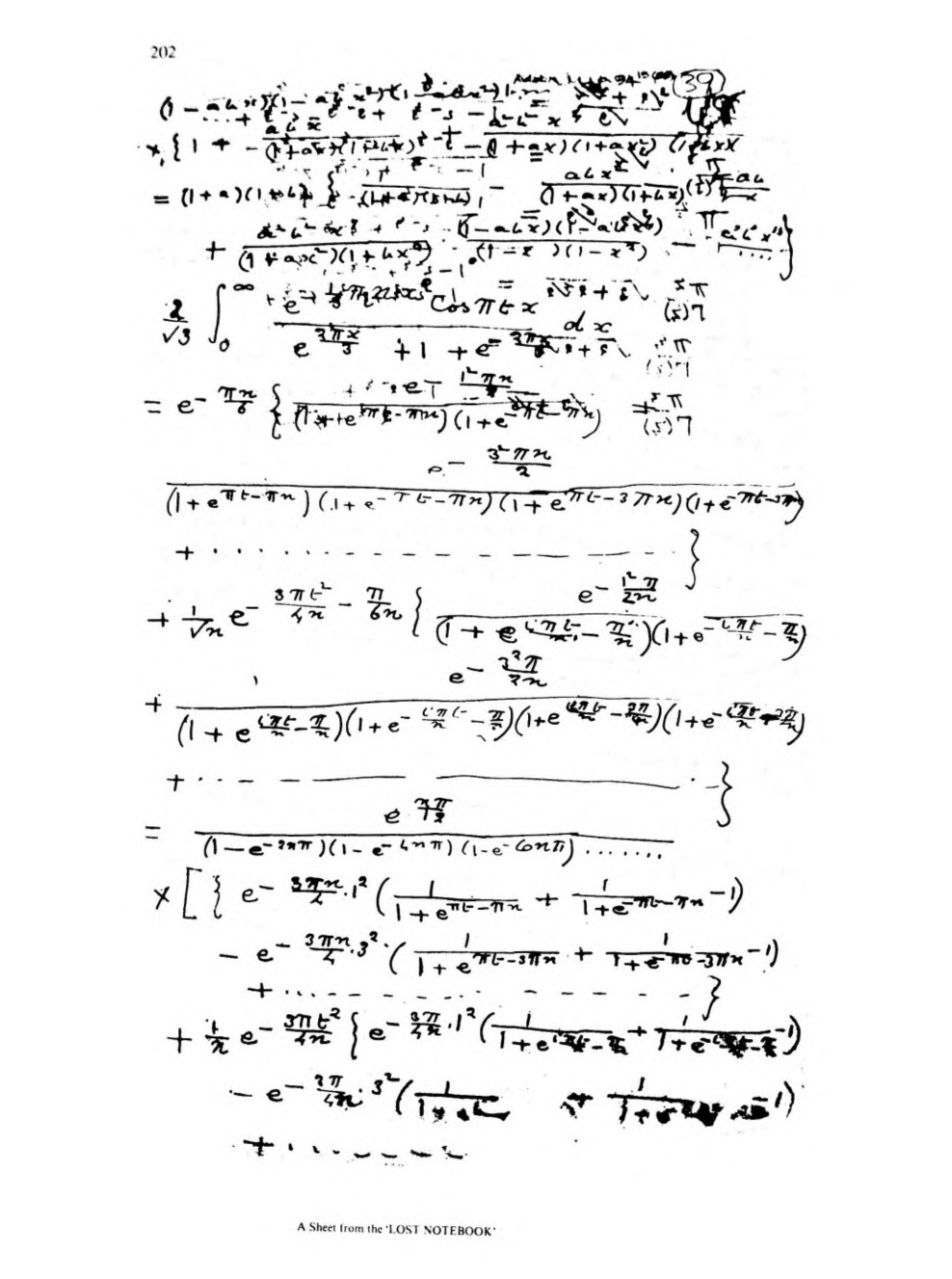
$$=\sqrt{\pi}\,e^{m^2}(1+ae^{-h^2+2imh})(1+ae^{-3h^2+2imh})\cdots$$

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(1+ae^{-2h^2+2hx})(1+ae^{-4h^2+2hx})\cdots$$
$$(1+be^{-2h^2-2hx})(1+be^{-4h^2-2hx})\cdots$$

$$=\sqrt{\pi}\,e^{m^2}\frac{(1-abe^{-2h^2})(1-abe^{-4h^2})\cdots}{(1-ae^{-h^2+2mh})(1-ae^{-3h^2+2mh}\cdots}$$
$$(1-be^{-h^2-2mh})(1-be^{-3h^2-2mh})\cdots$$

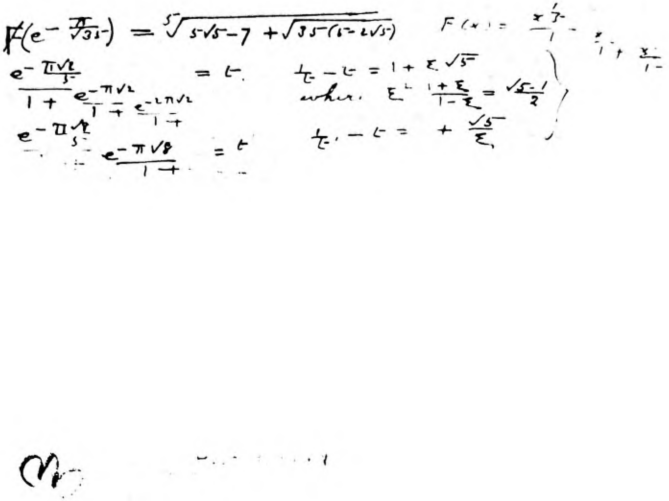
$$\int_{-\infty}^{\infty} e^{-x^2+2mx}\frac{dx}{(1-ae^{-h^2+2ihx})(1-ae^{-3h^2+2ihx})\cdots}$$
$$(1-be^{-h^2+2ihx})(1-be^{-3h^2+2ihx})\cdots$$

$$=\sqrt{\pi}\,e^{m^2}\frac{(1+ae^{-2h^2+2imh})(1+ae^{-4h^2+2imh})\cdots}{(1+be^{-2h^2-2imh})(1+be^{-4h^2-2imh})\cdots}$$
$$\overline{(1-abe^{-h^2})(1-abe^{-4h^2})(1-abe^{-6h^2}}$$

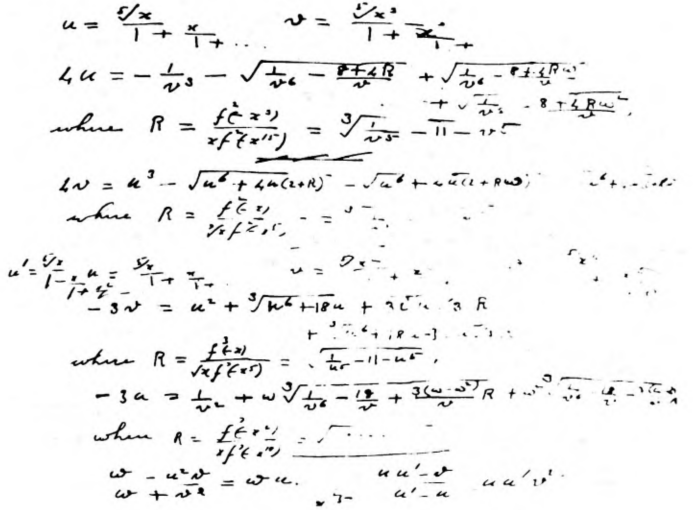


A Sheet from the 'LOST NOTEBOOK'

A Sheet from the 'LOST NOTEBOOK'



A Sheet from the 'LOST NOTEBOOK'



A Sheet from the 'LOST NOTEBOOK'

$$\frac{\phi(x^{\frac{1}{7}})}{\phi(x^7)} \quad 1 + u + v + w,$$

$$p = uvw = 8\,\frac{x^3(1+x)(1+x^3)(1+x^2)(1+x^7)\cdots}{\{(1+x^7)(1+x^{14})(1+x^{21})\cdots\}^7}$$

$$\frac{\phi^8(x)}{\phi^8(x^7)} - (2+5p)\frac{\phi^4(x)}{\phi^4(x^7)} + (1-p)^3 = 0$$

$$u = \left(\frac{\alpha^2\beta}{\gamma}\right)^{\frac{1}{7}}, \quad v = \left(\frac{\beta^2\gamma}{\alpha}\right)^{\frac{1}{7}}, \quad w = \left(\frac{\gamma^2\alpha}{\beta}\right)^{\frac{1}{7}}$$

where α, β, γ are the roots of the equation

$$\xi^3 + 2\xi^2\left\{1 + 3p - \frac{\phi^4(x)}{\phi^4(x^7)}\right\} + \xi p^2(p+4) - p^4 = 0$$

eg $\phi(e^{-7\pi\sqrt{7}}) = 7^{\frac{3}{8}}\phi(e^{-\pi\sqrt{7}})\{1 + (\ )^{\frac{1}{7}} + (\ )^{\frac{2}{7}} + (\ )^{\frac{3}{7}}\}^{\frac{1}{8}}$.

$$t = \frac{\sqrt{x}}{1+} \frac{x}{1+} \frac{x^2}{1+}$$

$$\frac{1}{\sqrt{t}} + \frac{\sqrt{5}-1}{2}\sqrt{t} = \frac{1}{\sqrt{x}}\sqrt{\frac{(1-x)(1-x^4)(1-x^5)\cdots}{(1-x^2)(1-x^3)(1-x^6)\cdots}}$$

$$\times \frac{1}{(1-\frac{\sqrt{5}-1}{2}\sqrt{x}+\sqrt{x^2})(1-\frac{\sqrt{5}-1}{2}\sqrt{x^2}+\sqrt{x^4})\cdots}$$

$$\frac{1}{\sqrt{t}} - \frac{\sqrt{5}+1}{2}\sqrt{t} = \frac{1}{\sqrt{x}}\sqrt{\frac{(1-x)(1-x^4)(1-x^5)\cdots}{(1-x^2)(1-x^3)(1-x^6)\cdots}}$$

$$\times \frac{1}{(1+\frac{\sqrt{5}+1}{2}\sqrt{x}+\sqrt{x^2})(1+\frac{\sqrt{5}+1}{2}\sqrt{x^2}+\sqrt{x^4})\cdots}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 + \left(\frac{\sqrt{5}-1}{2}\sqrt{t}\right)^5 = \frac{1}{\sqrt{x}}\sqrt{\frac{(1-x)(1-x^2)\cdots}{(1-x^5)(1-x^{10})\cdots}}$$

$$\times\left\{\frac{1}{(1-\frac{\sqrt{5}-1}{2}x+x^2)(1-\frac{\sqrt{5}-1}{2}x^2+x^4)\cdots}\right\}^5$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\frac{\sqrt{5}+1}{2}\sqrt{t}\right)^5 = \frac{1}{\sqrt{x}}\sqrt{\frac{(1-x)(1-x^2)\cdots}{(1-x^5)(1-x^{10})\cdots}}$$

$$\times\left\{\frac{1}{(1+\frac{\sqrt{5}+1}{2}x+x^2)(1+\frac{\sqrt{5}+1}{2}x^2+x^4)\cdots}\right\}^5$$

$$\frac{f(-\lambda x, -x^4)}{f(-x, -x^3)} f(-\lambda x^3) = f(-\lambda^2 x^3 - \lambda x^6) + x f(-\lambda, -\lambda^2 x^3)$$

$$P - Q = 1 + \frac{f(-x^{\frac{1}{2}}, -\lambda x^{\frac{5}{2}})}{x^{\frac{1}{2}} f(-\lambda^{10} x^5, -\lambda^{15} x^{10})}, \quad PQ = 1 - \frac{f(\lambda, x^2) f(-\lambda^2 x, -\lambda^3 x^2)}{f(-\lambda^{10} x^5, -\lambda^{15} x^{10})}$$

$$P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-x, -\lambda^5 x^4) f(-\lambda^2 x, -\lambda^3 x^2)}{f(-\lambda^{10} x^5, -\lambda^{15} x^{10})}$$

$$\frac{\phi(-x^3)}{\phi(-x)} = e^{2\sqrt{3}\int_0^x [\psi(t)\psi(t^3)]^2 \frac{dt}{t}}; \quad \sqrt{x}\frac{\psi(x^3)}{\psi(x)} = e^{\frac{1}{2}\int \left(\frac{\phi(x)\phi(x^3)}{x}\right)^2 dx}$$

$$\frac{\psi(-x)}{\psi(x)} = e^{-2\sqrt{3}\int_0^x \psi^3(t^2)\, dt} \qquad \frac{\psi(-x)}{\psi(x)} \cdot x^{\frac{1}{4}} e^{-\frac{1}{4}\int \frac{\phi^3(x)}{x} dx}$$

$$x\{(1-x)(1-x^2)\cdots\}^{24} = \sum_{n=1}^{\infty} D(n) x^n$$

$$\sum \frac{D(n)}{n^s} = \frac{1}{1+5^{1-s}} \prod_p \frac{1}{1 - B(p) p^{-s} + p^{2-3s}},$$

where p is a prime except 5. $\quad B(2) = 2^{-2}, \; B(3) = 2$

$$|B(p)| \le K p^{\frac{1}{2}} \qquad B(5) = -5, \; B(7) = 6$$

$$x^{\frac{1}{4}} \frac{(1-x)(1-x^4)^4 (1-x^7)^7 \cdots}{(1-x^2)^2 (1-x^3)^3 (1-x^6)^6 \cdots} = e^{-c - \frac{1}{2}\int_x^1 \frac{\{(1-t)(1-t^2)\cdots\}^9}{\{(1-t^2)^4(1-t^6)\cdots\}^3} \frac{dt}{t}}$$

$$c = \frac{3\sqrt{3}}{4\pi}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots\right) \; ??$$

$$\left(\frac{1-x\omega}{1-x\omega^2}\right)\left(\frac{1-x^2\omega}{1-x^2\omega^2}\right)^2 \left(\frac{1-x^3\omega}{1-x^3\omega^2}\right)^3 \left(\frac{1-x^4\omega}{1-x^4\omega^2}\right)^4 \cdots$$

$$= e^{-(\omega-\omega^2)\int_0^x \frac{\{(1-t^2)(1-t^4)(1-t^6)\cdots\}^9}{\{(1-t)(1-t^2)(1-t^3)\cdots\}^3} dt}$$

where $\omega^3 = 1$.

$p(n)$:— 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, ...

$A_5 = 0$, $A_{10} = 0$, $n \equiv 1, 2 \pmod 5$. $A_7 = 0$, $A_{14} = 0$, $n \equiv 1, 3, 4, \pmod 7$.
$A_{11} = 0$, $n \equiv 1, 2, 3, 5, 8 \pmod{11}$. $A_{13} = 0$, $n \equiv 2, 3, 5, 7, 9, 10 \pmod{13}$,
$A_{16} = 0$, $n \equiv 0 \pmod 2$. $A_{17} = 0$, $n \equiv 1, 3, 4, 6, 7, 9, 13, 14 \pmod{17}$.

122, 256, 276, 302, 332, 452, 536, 652, 722, 932,
1102, 1432, 1446, 1466, 1816, 1842, 1856, 1992, 1996.

$$t = \sqrt[5]{x}\cdot\frac{(1+x^5)(1+x^{10})(1+x^{15})\cdots}{(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots}$$

$$\frac{\sqrt[5]{x}}{1+\frac{x}{1+\cdots}} = \frac{1}{4t}\Big\{(1+t\cdot\tfrac{\sqrt5+1}{2})\sqrt{1-t} - \sqrt{(1-t)(1+t\cdot\tfrac{\sqrt5+1}{2})^2 - 2t(\sqrt5+1)}\Big\}$$
$$\Big\{-(1-t\cdot\tfrac{\sqrt5-1}{2})\sqrt{1-t} + \sqrt{(1-t)(1-t\cdot\tfrac{\sqrt5-1}{2})^2 + 2t(\sqrt5-1)}\Big\}$$

$$\frac{\sqrt[5]{x^2}}{1+\frac{x^2}{1+\cdots}} = \frac{1}{4t^2}\Big\{(1-t\cdot\tfrac{\sqrt5+1}{2})\sqrt{1-t} - \sqrt{(1-t)(1+t\cdot\tfrac{\sqrt5+1}{2})^2 - 2t(\sqrt5+1)}\Big\}$$
$$\Big\{-(1+t\cdot\tfrac{\sqrt5-1}{2})\sqrt{1-t} + \sqrt{(1-t)(1-t\cdot\tfrac{\sqrt5-1}{2})^2 + 2t(\sqrt5-1)}\Big\}$$

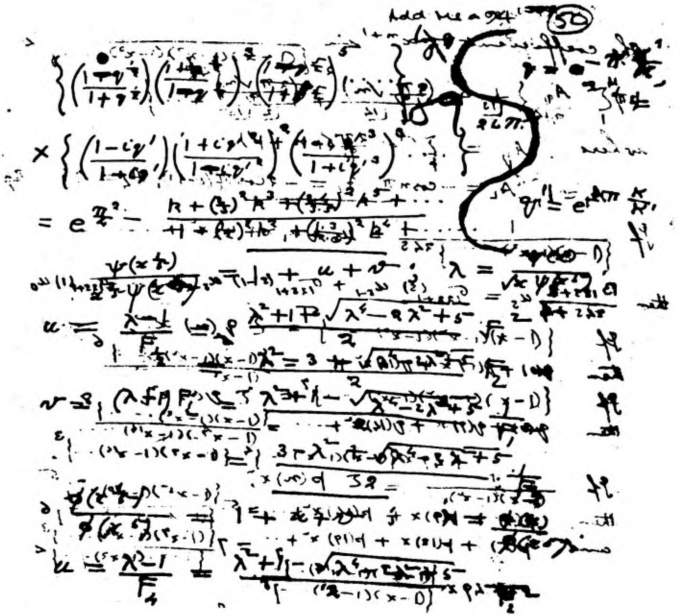
$$t = \sqrt[5]{x}\cdot\frac{(1+x^5)(1+x^{10})\cdots}{(1+x)(1+x^2)\cdots}$$

$$\Big\{\frac{\sqrt[5]{x}}{1+\frac{x}{1+\cdots}}\Big\}^5 = \frac{1}{4t^6}\Big[\Big\{1-t^6\big(\tfrac{\sqrt5-1}{2}\big)^2\Big\}\sqrt{1-t^6\big(\tfrac{\sqrt5-1}{2}\big)^6} - \Big\{1-t^6\big(\tfrac{\sqrt5-1}{2}\big)^5\Big\}\sqrt{1-t^6}\Big]$$
$$\times\Big[-\Big\{1-t^6\big(\tfrac{\sqrt5+1}{2}\big)^2\Big\}\sqrt{1-t^6\big(\tfrac{\sqrt5+1}{2}\big)^6} + \Big\{1+t^6\big(\tfrac{\sqrt5+1}{2}\big)^5\Big\}\sqrt{1-t^6}\Big]$$

$$= u v^2\,;\quad \Big\{\frac{\sqrt[5]{u}}{1+\frac{u}{1+\cdots}}\Big\}^5 = \frac{u^2}{v}, \text{ where}$$

$$u = \frac{1}{4t^6}\Big\{\sqrt{1-t^6} - \sqrt{1-t^6\big(\tfrac{\sqrt5+1}{2}\big)^6}\Big\}\Big\{\sqrt{1-t^6\big(\tfrac{\sqrt5-1}{2}\big)^6} - \sqrt{1-t^6}\Big\}$$

$$v = \frac{1-u}{1+u} = \frac{1}{4}\Big\{\sqrt{\big(\tfrac{\sqrt5+1}{2}\big)^6 - t^6} - \sqrt{1-t^6}\Big\}\Big\{\sqrt{\big(\tfrac{\sqrt5-1}{2}\big)^6 - t^6} + \sqrt{1-t^6}\Big\}$$



A Sheet from the 'LOST NOTEBOOK'

$$1-\alpha' = \frac{1-\alpha}{1+\alpha} \qquad \alpha = \frac{1-\alpha'}{1+\alpha'}$$

$$F(e^{-\frac{\pi}{\sqrt{5}}}) = \sqrt[5]{\frac{\sqrt{5}-1}{2}\left(\left(\frac{\sqrt{5}-1}{2}\right)^5 + \sqrt{1+\left(\frac{\sqrt{5}-1}{2}\right)^{10}}\right)}$$

$$\begin{cases} F(e^{-\frac{\pi}{\sqrt{15}}}) = \\ F(e^{-\pi\sqrt{\frac{3}{5}}}) = \\ F(e^{-\pi\sqrt{15}}) \end{cases}$$

$$\begin{cases} F(e^{-\frac{\pi}{\sqrt{35}}}) = \sqrt[5]{5\sqrt{5}-7+\sqrt{35(5-2\sqrt{5})}} \\ F(e^{-\pi\sqrt{\frac{7}{5}}}) = \sqrt[5]{-(7+5\sqrt{5})+\sqrt{35(5+2\sqrt{5})}} \end{cases}$$

$$\begin{cases} F(e^{-\frac{\pi}{\sqrt{105}}}) = \\ F(e^{-\pi\sqrt{\frac{21}{5}}}) = \end{cases}$$

$$\begin{cases} F(e^{-\frac{\pi}{\sqrt{55}}}) = \\ F(e^{-\pi\sqrt{\frac{11}{5}}}) = \end{cases}$$

$$F(e^{-\frac{\pi}{\sqrt{165}}})$$

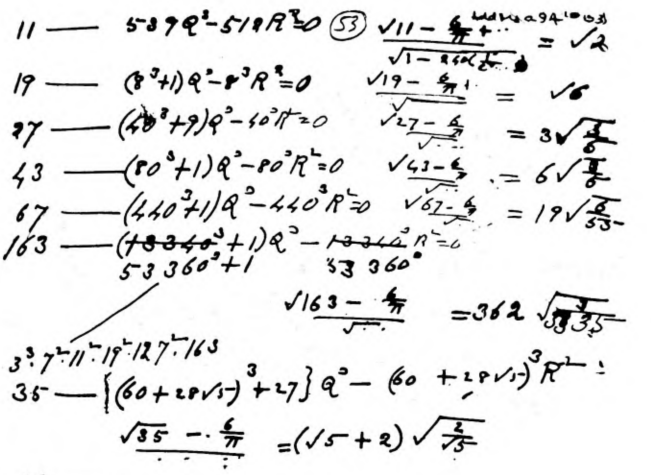
$$F(e^{-\pi\sqrt{\frac{33}{5}}})$$

$$F(e^{-\frac{\pi}{\sqrt{77}}})$$

$$F(e^{-\pi\sqrt{\frac{11}{7}}}) .$$

And $\frac{1}{\pi} = 0.318309\ldots$

(24)


11 —— $5\cdot 3^2 Q^2 - 5\cdot 1^2 R^2 = 0$ (53) $\sqrt{11 - \frac{6}{\pi}} = \sqrt{2}$
19 —— $(8^3+1)Q^2 - 8^3 R^2 = 0$ $\sqrt{19 - \frac{6}{\pi}} = \sqrt{6}$
27 —— $(40^3+9)Q^2 - 40^3 R^2 = 0$ $\sqrt{27 - \frac{6}{\pi}} = 3\sqrt{\frac{2}{6}}$
43 —— $(80^3+1)Q^2 - 80^3 R^2 = 0$ $\sqrt{43 - \frac{6}{\pi}} = 6\sqrt{\frac{2}{6}}$
67 —— $(440^3+1)Q^2 - 440^3 R^2 = 0$ $\sqrt{67 - \frac{6}{\pi}} = 19\sqrt{\frac{6}{53}}$
163 —— $(5\cdot 360^3+1)Q^2 - 33\,360^2 R^2 = 0$
$\sqrt{163 - \frac{6}{\pi}} = 362\sqrt{\frac{3}{33\,35}}$
$3^3 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 127^2 \cdot 163$
35 —— $\{(60+28\sqrt{5})^3 + 27\}Q^2 - (60+28\sqrt{5})^3 R^2 = 0$
$\sqrt{35 - \frac{6}{\pi}} = (\sqrt{5}+2)\sqrt{\frac{2}{5}}$
51 ——

$$\frac{e^{\frac{\pi}{2}\sqrt{\frac{23}{3}}}}{3\sqrt{3}} \cdot \left\{ (1+e^{-\pi\sqrt{3}})(1-e^{-3\pi\sqrt{3}})(1-e^{-5\pi\sqrt{3}}) \right\}^6 = \lambda_{\bullet}$$

$\lambda_1 = 1$, $\lambda_9 = 3$, $\lambda_{17} = 4+\sqrt{17}$

$\lambda_{25} = (2+\sqrt{5})^2$, $\lambda_{33} = 18+8\sqrt{33}$, $\lambda_{41} = 22+5\sqrt{41}$

$\lambda_{49} = 55+12\sqrt{21}$, $\lambda_{57} =$ $\lambda_{65} =$

$\lambda_{73} =$ $\lambda_{81} =$ λ_{89} λ_{97}

~~$\lambda_{121} =$~~ $\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}} \right)^6$

$\lambda_{97} = \left(\sqrt{\frac{17+\sqrt{97}}{8}} + \sqrt{\frac{9+\sqrt{97}}{8}} \right)^6$

$\lambda_{121} = \left(\frac{3\sqrt{3}+\sqrt{11}}{4} + \sqrt{\frac{11+3\sqrt{33}}{8}} \right)^6$

λ_{89} ~~$\lambda_{145} =$~~ $= 500 + 53\sqrt{89}$

$\lambda_{169} =$

$\lambda_{193} =$

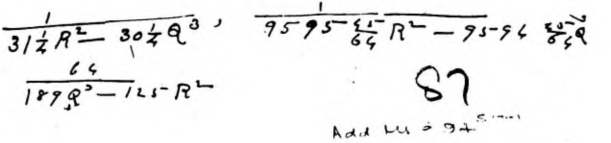
λ_{217}

λ_{241}

λ_{265}

λ_{289}

$\lambda_{361} =$

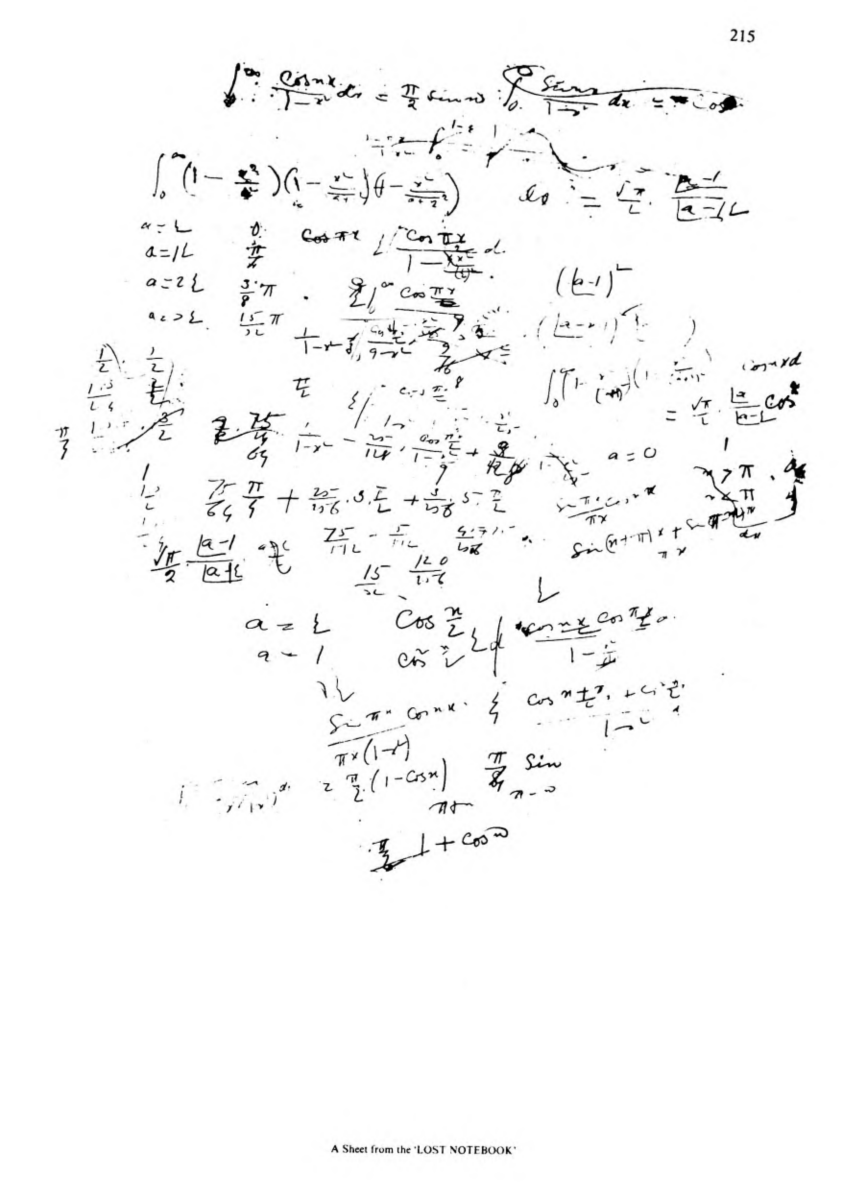


A Sheet from the 'LOST NOTEBOOK'

$$\int_0^\infty \frac{dx}{\Gamma(a+x)\Gamma(a-x)\Gamma(b+x)\Gamma(b-x)} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(a+b-\frac{1}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma(a+b)\Gamma\left(a+b-\frac{1}{2}\right)}$$

$$\int_0^\infty \frac{dx}{\Gamma(a+x)\Gamma(a-x)} = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\Gamma(a)\Gamma\left(a-\frac{1}{2}\right)}$$

A Sheet from the 'LOST NOTEBOOK'



A Sheet from the 'LOST NOTEBOOK'

Loose Papers (on Reciprocal Functions, Approximate Summations of Series Involving Prime Numbers, Euler Products for Dirichlet Series and Forty Identities etc.)

[Copied from the Loose Papers]

I If $\int_0^\infty f(x)\sin nx\,dx = \phi(n)$ and $\int_0^\infty f(x)e^{-nx}dx = \psi(n)$

then

(i) $\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty \psi(x)\cos nx\,dx$.

(ii) $\int_0^\infty \phi\left(\frac{t}{n}\right)e^{-nt}dt = -S^*\int_0^\infty \psi\left(\frac{t}{n}\right)\cos nx\,dx$.

* Summable.

Proof of (i) $\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty\int_0^\infty f(z)e^{-nx}\sin xz\,dx\,dz$

$= \int_0^\infty\int_0^\infty f(z)e^{-xz}\cos nx\,dx\,dz$

$= \int_0^\infty \psi(z)\cos nx\,dx$

(ii) $\int_0^\infty \phi\left(\frac{t}{n}\right)e^{-nt}dx = \int_0^\infty\int_0^\infty f(z)e^{-nz}\sin\frac{x}{n}\,dx\,dz$

$= \int_0^\infty\int_0^\infty n f(nz)e^{-nx}\sin z\,dx\,dz$

$= -\frac{d}{dn}\int_0^\infty\int_0^\infty f(xz)e^{-nx}\sin z\,dx\,dz$

$= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{z}{n}\right)\frac{\sin z}{z}\,dz$

$= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{t}{n}\right)\frac{\sin nx}{x}\,dx$

$= -S^*\int_0^\infty \psi\left(\frac{t}{n}\right)\cos nx\,dx$.

Similarly we can show that

II If $\int_0^\infty f(x)\cos nx\,dx = \phi(n)$ and $\int_0^\infty f(t)e^{-nt}dx = \psi(n)$

then

(i) $\int_0^\infty \phi(n)e^{-nx}dx = \int_0^\infty \psi(x)\sin nx\,dx$

(ii) $\int_0^\infty \phi\left(\frac{t}{n}\right)e^{-nx}dx = S\int_0^\infty \psi\left(\frac{t}{n}\right)\sin nx\,dx$

In I and II suppose that $f(x)$ is a reciprocal function of itself so that $f(x) = \sqrt{\frac{x}{n}}\,\phi(x)$

Hence we have

$\int_0^\infty \phi(x)e^{-nx}dx = \sqrt{\frac{n}{x}}\int_0^\infty f(x)e^{-nx}dx = \sqrt{\frac{n}{x}}\,\psi(x)$

Thus (i) of I and II reduce to III and IV

III If $\int_0^\infty \phi(x)\sin nx\,dx = \sqrt{\left(\frac{\pi}{2}\right)}\phi(n)$ and $\int_0^\infty \psi(x)e^{-nx}dx = \psi(n)$

then $\int_0^\infty \psi(x)\cos nx\,dx = \sqrt{\left(\frac{\pi}{2}\right)}\psi(n)$

IV If $\int_0^\infty \phi(x)\cos nx\,dx = \sqrt{\left(\frac{\pi}{2}\right)}\phi(n)$ and $\int_0^\infty \psi(x)e^{-nx}dx = \psi(n)$.

then $\int_0^\infty \psi(x)\sin nx\,dx = \sqrt{\left(\frac{\pi}{2}\right)}\psi(n)$.

The formulae III and IV enable us to find a number of

reciprocal functions of the first and second kind out of one reciprocal function. For example we have

$$(1)\quad \int_0^\infty \left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)\sin 2\pi n x\,dx=\frac{1}{2}\left(\frac{1}{e^{2\pi n}-1}-\frac{1}{2\pi n}\right)$$

Also we have

$$(2)\quad \int_0^\infty \left(\frac{1}{e^{2\pi x}-1}-\frac{1}{2\pi x}\right)e^{-2\pi n x}\,dx=\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\log n$$

Hence

$$(3)\quad \int_0^\infty \left\{\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\log n\right\}\cos 2\pi n x\,dx=\frac{1}{2}\left\{\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\log n\right\}$$

Again we have

$$(4)\quad \int_0^\infty \frac{\cos\frac{1}{2}\pi n x}{\cosh\frac{1}{2}\pi x}\,dx=\frac{1}{\cosh\frac{1}{2}\pi n}$$

and

$$(5)\quad \int_0^\infty \frac{e^{-\frac{1}{2}\pi n x}}{\cosh\frac{1}{2}\pi x}\,dx=\frac{4}{\pi}\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right)$$

Hence

$$(6)\quad \int_0^\infty \left(\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+5}-\cdots\right)\sin\tfrac{1}{2}\pi n x\,dx=\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots$$

By applying the theory of reciprocal functions to III and IV, we can obtain a number of curious results as the following:

It can be deduced from (3) that if $\alpha\beta=1$ and

$$\phi(x)=\frac{\Gamma'(x)}{\Gamma(x)}+\frac{1}{2x}-\log x,$$

then

$$(7)\quad \sqrt{\alpha}\left\{\frac{\gamma-\log 2\pi\alpha}{2\alpha}+\phi(\alpha)+\phi(2\alpha)+\phi(3\alpha)+\cdots\right\}$$

$$=\sqrt{\beta}\left\{\frac{\gamma-\log 2\pi\beta}{2\beta}+\phi(\beta)+\phi(2\beta)+\phi(3\beta)+\cdots\right\}$$

$$=-\frac{1}{\pi\sqrt{\pi}}\int_0^\infty \left|\Gamma\left(\tfrac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\,dt.$$

I If $\int_0^\infty \phi(x)\cos nx\,dx=\sqrt{\left(\frac{\pi}{2}\right)}\psi(n)$ and $\int_0^\infty e^{-nx}\phi(x)\,dx=\psi(n)$

then $\int_0^\infty \psi(x)\sin nx\,dx=\sqrt{\left(\frac{\pi}{2}\right)}\phi(n)$

II If $\int_0^\infty \phi(x)\sin nx\,dx=\sqrt{\frac{\pi}{2}}\psi(n)$ and $\int_0^\infty e^{-nx}\psi(x)\,dx=\phi(n)$,

Then $\int_0^\infty \psi(x)\cos nx\,dx=\sqrt{\frac{\pi}{2}}\phi(n)$.

Example for I. We have $\int_0^\infty \frac{\cos nx}{\cosh x\sqrt{\frac{\pi}{2}}}\,dx=\frac{\sqrt{\frac{\pi}{2}}}{\cosh n\sqrt{\frac{\pi}{2}}}$

and $\int_0^\infty \frac{e^{-nx}}{\cosh x\sqrt{\frac{\pi}{2}}}\,dx=2\left(\frac{1}{n+\sqrt{\frac{\pi}{2}}}-\frac{1}{n+3\sqrt{\frac{\pi}{2}}}+\frac{1}{n+5\sqrt{\frac{\pi}{2}}}-\cdots\right)$

Hence $\int_0^\infty \left(\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+5}-\cdots\right)\sin\frac{1}{2}\pi n x\,dx=\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots$,

Example for II. We have $\int_0^\infty \left(\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}}\right)\sin nx\,dx=\sqrt{\frac{\pi}{2}}\left\{e^{-n\sqrt{2\pi}}-\frac{1}{n\sqrt{2\pi}}\right\}$

and $\int_0^\infty \left(\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}}\right)e^{-nx}\,dx=\sqrt{2\pi}\left\{\gamma+\log\frac{n}{\sqrt{2\pi}}-\frac{\Gamma'\left(1+\frac{n}{\sqrt{2\pi}}\right)}{\Gamma\left(1+\frac{n}{\sqrt{2\pi}}\right)}\right\}$

Hence

$$\int_0^\infty \left\{\gamma+\log x-\frac{\Gamma'(1+x)}{\Gamma(1+x)}\right\}\cos 2\pi n x\,dx=\frac{1}{2}\left\{\gamma+\log n-\frac{\Gamma'(1+n)}{\Gamma(1+n)}\right\}.$$

$$\phi_\omega(t) = \int_0^\infty \frac{\cos\tfrac{\pi t x}{\omega}}{\cosh \pi x} e^{-\pi\omega x^2} dx \;;\; \psi_\omega(t) = \int_0^\infty \frac{\sin\tfrac{\pi t x}{\omega}}{\sinh \pi x} e^{-\pi\omega x^2} dx$$

$$\phi_\omega(t) = \phi_\omega(-t) \;;\; \psi_\omega(t) = -\psi_\omega(-t).$$

$$\phi_\omega(t) = 2\int_0^\infty\int_0^\infty \frac{\cos 2\pi x z}{\cosh \pi z} \cos \pi t x \, e^{-\pi\omega x^2} dx\, dz$$

$$= \frac{e^{-\frac{\pi t^2}{4\omega}}}{\sqrt{\omega}} \int_0^\infty \frac{\cosh \frac{\pi t x}{2\omega}}{\cosh \pi x} e^{-\frac{\pi}{4\omega} x^2} dx$$

$$\phi_\omega(t) = \frac{1}{\sqrt{\omega}} e^{-\frac{\pi t^2}{4\omega}} \phi_{\frac{1}{4\omega}}\left(\frac{it}{2\omega}\right)$$

$$\phi_\omega(t+\omega) = \frac{1}{\sqrt{\omega}} e^{-\frac{\pi(t+\omega)^2}{4\omega}} \int_0^\infty \frac{\cosh\frac{\pi t x}{2\omega}\cosh \pi x + \sinh\frac{\pi t}{2\omega}\sinh \pi x}{\cosh \pi x} e^{-\frac{\pi}{4\omega}x^2} dx$$

$$= \frac{1}{\sqrt{\omega}} e^{-\frac{\pi(t+\omega)^2}{4\omega}} \left\{ \frac{1}{2}\sqrt{\omega}\, e^{\frac{\pi t^2}{4\omega}} + 2\int_0^\infty\int_0^\infty \frac{\sin 2\pi x z}{\sinh \pi z} \sinh\tfrac{\pi t x}{2} e^{-\frac{\pi}{4\omega}x^2} dx\, dz \right\}$$

$$e^{\frac{\pi t^2}{4\omega}}\left\{\frac{1}{2} + \psi_\omega(t)\right\} = e^{\frac{\pi(t+\omega)^2}{4\omega}} \phi_\omega(t+\omega)$$

$$\frac{1}{2} + \psi_\omega(t+i) = \frac{i}{\sqrt{\omega}} e^{-\frac{\pi t^2}{4\omega}} \left\{ \frac{1}{2} - \psi_{\frac{i}{2\omega}}\left(\frac{it}{2\omega} + i\right)\right\}$$

$$\phi_\omega(i) = \frac{1}{2\sqrt{\omega}} \;;\; \psi_\omega(i) = \frac{i}{2\sqrt{\omega}} \;;\; \phi_\omega(\omega) = \frac{1}{2} e^{-\frac{\pi\omega}{4}} ; \frac{1}{2} - \psi_\omega(\omega) = e^{-\frac{\pi\omega}{4}} \phi_\omega(0);$$

$$\phi_\omega(\omega \pm i) = \left(\frac{1}{2\sqrt{\omega}} \mp \frac{i}{2}\right) e^{-\frac{\pi\omega}{4}}, \psi_\omega(\omega \pm i) = \frac{1}{2} \pm \frac{i}{2\sqrt{\omega}} e^{-\frac{\pi\omega}{4}}; \phi_\omega\left(\tfrac{1}{2}\omega\right) + \psi_\omega\left(\tfrac{1}{2}\omega\right) = \frac{1}{2}.$$

$$\phi_\omega(t+i) + \phi_\omega(t-i) = \frac{1}{\sqrt{\omega}} e^{-\frac{\pi t^2}{4\omega}} \;;\; \psi_\omega(t+i) - \psi_\omega(t-i) = \frac{i}{\sqrt{\omega}} e^{-\frac{\pi t^2}{4\omega}}$$

$$e^{\frac{\pi(t+\omega)^2}{4\omega}} \phi_\omega(t+\omega) + e^{\frac{\pi(t-\omega)^2}{4\omega}} \phi_\omega(t-\omega) = e^{\frac{\pi t^2}{4\omega}};$$

$$e^{\frac{\pi(t+\omega)^2}{4\omega}}\left\{\frac{1}{2} - \psi_\omega(t+\omega)\right\} = e^{\frac{\pi(t-\omega)^2}{4\omega}}\left\{\frac{1}{2} + \psi_\omega(t-\omega)\right\}$$

$$\phi_\omega(t) + (-1)^{\cdots}\phi_\omega(t+2\omega i) = \frac{1}{\sqrt{\omega}}\left\{e^{-\frac{\pi}{4\omega}(t+i)^2} - e^{-\frac{\pi}{4\omega}(t+3i)^2} + \cdots \text{ to } n \text{ terms}\right\}$$

$$\psi_\omega(t) - \psi_\omega(t+2\omega i) = -\frac{i}{\sqrt{\omega}}\left\{e^{-\frac{\pi}{4\omega}(t+i)^2} + e^{-\frac{\pi}{4\omega}(t+3i)^2} + \cdots \text{ to } n \text{ terms}\right\}$$

$$e^{\frac{\pi t^2}{4\omega}}\phi_\omega(t) + (-1)^{\cdots} e^{\frac{\pi(t+2\omega)^2}{4\omega}} \phi_\omega(t+2\omega) = e^{\frac{\pi}{4\omega}(t+\omega)^2} - e^{\frac{\pi}{4\omega}(t+3\omega)^2} + \cdots \text{ to } n \text{ terms}$$

$$e^{\frac{\pi t^2}{4\omega}}\left\{\frac{1}{2} + \psi_\omega(t)\right\} + (-1)^{\cdots} e^{\frac{\pi(t+2\omega)^2}{4\omega}}\left\{\frac{1}{2} + \psi_\omega(t+2\omega)\right\}$$
$$= e^{\frac{\pi}{4\omega}(t+2\omega)^2} - e^{\frac{\pi}{4\omega}(t+4\omega)^2} + \cdots \text{ to } n \text{ terms}.$$

If $s = t + 2m\omega \pm 2ni$, then

I $\phi_\omega(s) + (-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m(s+t)} \phi_\omega(t)$

$= e^{-\frac{\pi s^2}{4\omega}} \left\{ e^{\frac{\pi}{4\omega}(s-\omega)^2} - e^{\frac{\pi}{4\omega}(s-3\omega)^2} + e^{\frac{\pi}{4\omega}(s-5\omega)^2} + \dots \text{ to } m \text{ terms} \right\}$

$+ \frac{(-1)^{(m+1)(n+1)}}{\sqrt{i\omega}} e^{\frac{1}{2}\pi m(s+t)} \left\{ e^{-\frac{\pi}{4\omega}(t \pm i)^2} - e^{-\frac{\pi}{4\omega}(t \pm 3i)^2} + \dots \text{ to } n \text{ terms} \right\}$

II $\frac{1}{2} - \psi_\omega(s) + (-1)^{mn+m+n} e^{-\frac{1}{2}\pi m(s+t)} \left\{ \frac{1}{2} - \psi_\omega(t) \right\}$

$= e^{-\frac{\pi s^2}{4\omega}} \left\{ e^{\frac{\pi}{4\omega}(s-\omega)^2} - e^{\frac{\pi}{4\omega}(s-3\omega)^2} + \dots \text{ to } m \text{ terms} \right\}$

$\pm (-1)^{mn+m+n} \frac{i}{\sqrt{i\omega}} e^{-\frac{1}{2}\pi m(s+t)} \left\{ e^{-\frac{\pi}{4\omega}(t \pm i)^2} + e^{-\frac{\pi}{4\omega}(t \pm 3i)^2} + \dots \text{ to } n \text{ terms} \right\}$

If $t = m\omega \pm ni$, where m is odd and n is odd; m is even and n is odd; m is odd and n is even

$\phi_\omega(t) = \frac{1}{2} e^{-\frac{\pi t^2}{4\omega}} \left\{ e^{\frac{\pi}{4\omega}(t-\omega)^2} - e^{\frac{\pi}{4\omega}(t-3\omega)^2} + \dots \text{ to } m \text{ terms} \right\}$

$+ \frac{i}{2\sqrt{i\omega}} \left\{ e^{-\frac{\pi}{4\omega}(t \mp i)^2} - e^{-\frac{\pi}{4\omega}(t \mp 3i)^2} + \dots \text{ to } n \text{ terms} \right]$

If $t = m\omega \pm ni$ where m is odd and n is odd; m is even and n is odd; m is even and n is even

$\psi_\omega(t) = -\frac{1}{2} e^{-\frac{\pi t^2}{4\omega}} \left\{ e^{\frac{\pi}{4\omega}(t-2\omega)^2} - e^{\frac{\pi}{4\omega}(t-4\omega)^2} + \dots \text{ to } m \text{ terms} \right\}$

$\pm \frac{i}{2\sqrt{i\omega}} \left\{ e^{-\frac{\pi}{4\omega}(t \mp i)^2} + e^{-\frac{\pi}{4\omega}(t \mp 3i)^2} + \dots \text{ to } n \text{ terms} \right\}$

If $t = m\omega \pm ni$ then

$\phi_\omega(t) = e^{-\frac{1}{2}\pi m^2\omega} \left\{ \frac{1}{2}\left(\frac{1}{\sqrt{i\omega}} + e^{\mp \frac{1}{2} i\pi m n} \right) - \frac{1}{2}\pi m \right.$?

Reciprocal Functions

1 Let $\mathfrak{X}_1(s) = \int_0^\infty x^{s-1} \chi_1(x)\,dx$, $\mathfrak{X}_2(s) = \int_0^\infty x^{s-1} \chi_2(x)\,dx$.

The functions χ_1 and χ_2 are so related that

(1) $\quad \mathfrak{X}_1(s)\,\mathfrak{X}_2(1-s) = \lambda^2$

where λ is independent of s.

Then the two equations

(2) $\quad \begin{cases} \int_0^\infty \phi(x)\,\chi_1(xy)\,dx = \lambda\,\psi(y) \\ \int_0^\infty \psi(x)\,\chi_2(xy)\,dx = \lambda\,\phi(y) \end{cases}$

imply each other. From (1) we see that χ_1 and χ_2 may be interchanged and that

(3) $\quad \left(\frac{1}{2}\lambda\right)^2 = \int_0^\infty \chi_1(x^2)\,dx \int_0^\infty \chi_2(x^2)\,dx$.

$\chi_1(x)$ and $\chi_2(x)$ can be changed to $a_1\chi_1(cx)$ and $a_2\chi_2(cx)$. The λ must be changed to $\lambda N(a_1a_2/c)$. Now let

$$Z_1(s) = \int_0^\infty x^{s-1}\phi(x)\,dx \;;\; Z_2(s) = \int_0^\infty x^{s-1}\psi(x)\,dx.$$

Then we have

(4) $\quad \frac{Z_1(s)}{Z_2(1-s)} = \frac{\mathfrak{X}_2(s)}{\lambda} = \frac{\lambda}{\mathfrak{X}_1(1-s)}$

Now it is well known that the two equations

(5) $\quad \int_0^\infty x^{y-1} f(x)\,dx = F(y) \;;\; \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-y} F(y)\,dy = f(x)$

imply each other. Thus we see that if χ_1 is an arbitrary function, then χ_2 can be ascertained with the help of (4) and (5).

The equations (1) and (4) imply both the equations (2).

The well known examples are

(6) $\quad \begin{cases} \chi_1(x) = \chi_2(x) = \cos x \;;\; \lambda^2 = \frac{1}{2}\pi \\ \chi_1(x) = \chi_2(x) = \sin x \;;\; \lambda^2 = \frac{1}{2}\pi \end{cases}$

(7) $\quad \chi_1(x) = \chi_2(x) = J_\nu(x)\sqrt{x} \;;\; \lambda = 1,$

ν being any number whose real part is greater than -1.

(8) $\quad \chi_1(x) = \chi_2(x) = \frac{x^\nu}{1-x^2} \;;\; \lambda = \frac{\pi}{2}$

where ν is any integer. In the following examples ϕ and ψ are the same functions.

(9) $\quad \begin{cases} \int_0^\infty e^{-x^2}\cos 2xy\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-y^2}, \\ \int_0^\infty x\,e^{-x^2}\sin 2xy\,dx = \frac{1}{2}\sqrt{\pi}\,y\,e^{-y^2}. \end{cases}$

(10) $\quad \int_0^\infty x^{\nu+\frac{1}{2}} e^{-x^2} J_\nu(xy)\,dx = \sqrt{\frac{2}{\pi}}\,\Gamma(\nu+1)\,y^\nu e^{-y^2}$

(11) $\quad \int_0^\infty \frac{1}{1+x^2}\frac{dx}{1-y^2x^2} = \frac{\pi}{2}\frac{1}{1+y^2}$.

2. The construction of χ_1 and χ_2

If χ_1 is given it is theoretically (though not always) possible to find χ_2 with the help of (1) and (5). In this procedure χ_1 and χ_2 are generally neither the same function nor similar functions, nor both of them capable of finite expression at the same time. It is also extremely improbable that we can select functions like $\cos x$ and $\sin x$. We shall now proceed in a different way. It is always possible to choose a number of functions $X_1(s)$ and $X_2(s)$ which are either the same function or similar functions so that $X_1(s)X_2(1-s)$ is an absolute constant. Then by (5) we have

$$\begin{cases} \chi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} X_1(s)\,ds \\ \chi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} X_2(s)\,ds. \end{cases} \tag{12}$$

For instance suppose that

$$X_1(s) = X_2(s) = \Gamma(s)\cos\tfrac{1}{2}\pi s \text{ or } \Gamma(s)\sin\tfrac{1}{2}\pi s \tag{13}$$

so that $X_1(s)X_2(1-s)$ is a constant. Then from (12) we deduce that $\chi_1(x) = \chi_2(x) = \cos x$ or $\sin x$ by using the well known formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)\,ds = e^{-x}.$$

Suppose now that the $2g$ integers

$$0, 1, 2, 3, 4, \ldots 2g-1,$$

are divided into any group of g integers, say

$$a_1, a_2, \ldots a_g$$
$$b_1, b_2, \ldots b_g$$

and that

$$\begin{cases} X_1(s) = \Gamma(s)\prod_{r=1}^{g} \sin\frac{\pi(s+a_r)}{2g}, \\ X_2(s) = \Gamma(s)\prod_{r=1}^{g} \sin\frac{\pi(s-a_r+2g-1)}{2g} \end{cases} \tag{15}$$

so that $X_1(s)X_2(1-s)$ is a constant. Then we can easily find $\chi_1(s)$ and $\chi_2(s)$ with the help of (11) and (14). Here the number of choices of the a's and b's is

$$\frac{(2g)!}{(g!)^2} \tag{16}$$

But the number of solutions for χ_1 and χ_2 in this case (say belonging to class g includes the number of solutions belonging to the class δ when δ is a divisor of g. Hence eliminating all these extraneous solutions we find the number of ways belonging to class g is

$$\omega = \sum_{\delta}{}' \mu(\delta)\frac{(2g/\delta)!}{\{(g/\delta)!\}^2} \tag{17}$$

Thus we have

$g=1 \quad \omega = \frac{2!}{(1!)^2} = 2$

$g=2 \quad \omega = \frac{4!}{(2!)^2} - \frac{2!}{(1!)^2} = 4$

$g=3 \quad \omega = \frac{6!}{(3!)^2} - \frac{2!}{(1!)^2} = 18$

$g=4 \quad \omega = \frac{8!}{(4!)^2} - \frac{4!}{(2!)^2} = 64$

$g=5 \quad \omega = \frac{10!}{(5!)^2} - \frac{2!}{(1!)^2} = 250$

$g=6. \quad \omega = \frac{12!}{(6!)^2} - \frac{6!}{(3!)^2} - \frac{4!}{(2!)^2} + \frac{2!}{(1!)^2} = 900$

$g=7 \quad \omega = \frac{14!}{(7!)^2} - \frac{2!}{(1!)^2} = 3430$

$g=8 \quad \omega = \frac{16!}{(8!)^2} - \frac{8!}{(4!)^2} = 12800$

$g=9 \quad \omega = \frac{18!}{(9!)^2} - \frac{6!}{(3!)^2} = 48600$

$g=10 \quad \omega = \frac{20!}{(10!)^2} - \frac{10!}{(5!)^2} - \frac{4!}{(2!)^2} + \frac{2!}{(1!)^2} = 184500$

I shall give the solutions of χ_1 and χ_2 belonging to the first three cases.

$g=1$. Trigonometric functions only.

$g=2$ Trig. functions and e^{-x}.

$g=3$ — do — $e^{-x\sqrt{3}}$

$g=4$ — do — e^{-x} and $e^{-x\sqrt{2}}$

$g=5$ — do — e^{-2x} and $e^{-x(1+\sqrt{5})}$

$g=6$ — do — e^{-3}, e^{-2x} and $e^{-x\sqrt{3}}$

and so on.

3. The case in which $\chi_1(x) = \chi_2(x)$, $\phi(x) = \psi(x)$

Let $X(s) = \int_0^\infty x^{s-1} \chi(x)\, dx$,

and let the function $\chi(x)$ satisfy the relation

(18) $\qquad X(s)\, X(1-s) = \lambda^2$

where λ is independent of s. Then the two equations

(19) $\qquad \begin{cases} \int_0^\infty \phi(x) \chi(nx)\, dx = \lambda \psi(n) \\ \int_0^\infty \psi(x) \chi(nx)\, dx = \lambda \phi(n) \end{cases}$

imply each other. It follows from (3) that

(20) $\qquad \pm \lambda = \pm \int_0^\infty \chi(x^2)\, dx.$

We have already discussed about the construction of $\chi(x)$. We shall now consider the interesting case in which $\phi(x) = \psi(x)$. This is really two cases one in which λ has the positive sign and the other in which it has the negative sign. It follows from (4) that if

$$Z(s) = \int_0^\infty x^{s-1} \phi(x)\, dx$$

and

(21) $\qquad \frac{Z(s)}{Z(1-s)} = \frac{X(s)}{\lambda} = \frac{\lambda}{X(1-s)}$

then

(22) $\qquad \int_0^\infty \phi(x) \chi(nx)\, dx = \lambda\, \phi(n)$

Thus $\phi(x)$ is a positive reciprocal function of itself or a negative reciprocal function of itself according as λ is taken with positive or negative sign from (20). Suppose now that $T(s)$ is a function such that $T(s) = T(1-s)$

for instance Riemann's $\xi(s)$ or some such function) and also that a solution of $Z(s)$ is found from (21). Then we can replace $Z(s)$ by $Z(s)\,P(s)$ and get a new ϕ function satisfying (22) for every $P(s)$ we choose. The following methods will show that the construction of ϕ is much easier than that of χ in the preceding section.

Let $f(x)$ be an arbitrary function. Then if

$$\phi(x) = f(x) + \int_0^\infty f(t)\,\chi(xt)\,dt \tag{23}$$

it follows from (19) that

$$\int_0^\infty \phi(t)\,\chi(xt)\,dt = \lambda\,\phi(x)$$

Here λ may have any of the two values in (24).

Again if $f(x)$ be any function such that

$$f\left(\tfrac{1}{x}\right) = x\,f(x)$$

and if $\int_0^\infty \phi(x)\,\chi(xt)\,dx = \lambda\,\phi(t)$

then

$$\int_0^\infty F(x)\,\chi(xt)\,dx = \lambda\,F(t) \tag{24}$$

where $\int_\epsilon^{1/\epsilon} f(x)\,\phi(xt)\,dx = F(t)$

ϵ being any positive number including zero. This is very easy to prove. For example we have

$$\int_0^\infty F(x)\,\chi(xt)\,dx = \lambda\,F(t) \tag{25}$$

if $\int_\epsilon^{\frac{1}{\epsilon}} \frac{x^{\frac{1}{2}(ab-1)}}{(1+x^a)^{\frac{b}{2}}}\, e^{-\pi(x^v+x^{-v})}\,\phi(xt)\,dx = F(t)$

where ϵ is zero or a positive number

a and b are any two numbers.

real parts of u and v are positive and $\phi(t)$ is any reciprocal function as in (22).

The simplest cases of (25) are then

$$F(t) = \int_0^\infty \frac{\phi(xt)}{1+x}\,dx \tag{26}$$

$$F(t) = \int_0^\infty \frac{\phi(xt)}{\sqrt{(1+x^2)}}\,dx \tag{27}$$

$$F(t) = \int_0^\infty e^{-\pi(x^v+x^{-v})}\,\phi(xt^2)\,dx \tag{28}$$

For instance it follows from (10) that

$$\int_0^\infty F(t)\,J_\nu(xt)\sqrt{(xt)}\,dt = \tfrac{1}{2}\,F(x). \tag{29}$$

then $F(v) = v^{\nu+1}\int_\epsilon^{\frac{1}{\epsilon}} x^\nu\, e^{-(u+v)x^2 - \frac{u}{x^2}}\,dx.$

$g=1,\ \omega=2$

$\lambda=\sqrt{(\frac{1}{2}\pi)}$ $\quad \chi_1(x)=\chi_2(x)=\cos x$

$\lambda=\sqrt{(\frac{1}{2}\pi)}$ $\quad \chi_1(x)=\chi_2(x)=\sin x$

$g=2,\ \omega=4$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_1(x)=\cos x-\sin x+e^{-x}$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\sin x-\cos x+e^{-x}$

$\lambda=\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\cos x+\sin x+e^{-x}\\ \chi_2(x)=\cos x+\sin x-e^{-x}\end{cases}$

$g=3,\ \omega=18$

$\lambda=\frac{1}{2}\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\cos 2x-e^{-x\sqrt{3}}(\cos x+\sqrt{3}\sin x)$

$\lambda=\frac{1}{2}\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\sin 2x+e^{-x\sqrt{3}}(\sin x-\sqrt{3}\cos x)$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\cos 2x-\sqrt{3}\sin 2x+2e^{-x\sqrt{3}}\cos x$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\sin 2x+\sqrt{3}\cos 2x-2e^{-x\sqrt{3}}\sin x$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\cos 2x+\sqrt{3}\sin 2x-e^{-x\sqrt{3}}(\cos x-\sqrt{3}\sin x)$

$\lambda=\sqrt{\pi}$ $\quad \chi_1(x)=\chi_2(x)=\sin 2x-\sqrt{3}\cos 2x+e^{-x\sqrt{3}}(\sin x+\sqrt{3}\cos x)$

$\lambda=\frac{1}{2}\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\cos 2x+2e^{-x\sqrt{3}}\cos x\\ \chi_2(x)=\cos 2x-e^{-x\sqrt{3}}(\cos x-\sqrt{3}\sin x)\end{cases}$

$\lambda=\frac{1}{2}\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\sin 2x-2e^{-x\sqrt{3}}\sin x\\ \chi_2(x)=\sin 2x+e^{-x\sqrt{3}}(\sin x+\sqrt{3}\cos x)\end{cases}$

$\lambda=\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\cos 2x+\sqrt{3}\sin 2x+2e^{-x\sqrt{3}}\cos x\\ \chi_2(x)=\cos 2x+\sqrt{3}\sin 2x-e^{-x\sqrt{3}}(\cos x+\sqrt{3}\sin x)\end{cases}$

$\lambda=\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\sin 2x-\sqrt{3}\cos 2x-2e^{-x\sqrt{3}}\sin x\\ \chi_2(x)=\sin 2x-\sqrt{3}\cos 2x+e^{-x\sqrt{3}}(\sin x-\sqrt{3}\cos x)\end{cases}$

$\lambda=\sqrt{\pi}$ $\quad \begin{cases}\chi_1(x)=\cos 2x-\sqrt{3}\sin 2x-e^{-x\sqrt{3}}(\cos x+\sqrt{3}\sin x)\\ \chi_2(x)=\cos 2x-\sqrt{3}\sin 2x-e^{-x\sqrt{3}}(\cos x-\sqrt{3}\sin x)\end{cases}$

$\chi_1(x)=\sin 2x+\sqrt{3}\cos 2x+e^{-x\sqrt{3}}(\sin x+\sqrt{3}\cos x)$

$\chi_2(x)=\sin 2x+\sqrt{3}\cos 2x+e^{-x\sqrt{3}}(\sin x-\sqrt{3}\cos x)$

1. In this paper I consider approximate summations of series involving prime numbers. The approximate summation of the series

(1) $\phi(2)+\phi(3)+\phi(5)+\cdots+\phi(p)$

in terms of the number of terms is somewhat more regular and approximate than that in terms of p. Let p be the greatest prime not exceeding x, $\pi(x)$ the number of primes not exceeding x and also let

$$\vartheta(x)=\log 2+\log 3+\log 5+\cdots+\log p \quad \text{and} \quad \pi(x)=u.$$

Then it can easily be shown that

(2) $\pi(x)\log x-\vartheta(x)=\int_2^x \frac{\pi(t)}{t}\,dt$

Without assuming the prime number theorem we have

(3) $\log x = \log u+\log\log u+O(1)$

It follows that

(4) $\log 2+\log 3+\log 5+\cdots+\log u \text{ terms} = u\log u+u\log\log u+O(u)$

This is obtained by purely elementary reasoning. But

(5) $\vartheta(p)\sim p$

is as deep as the prime number theorem.

2. Let us now assume the Riemann hypothesis and express $\vartheta(x)$ in terms of u. We have

(6) $\vartheta(x)=x-\sqrt{x}-\sum\frac{x^\rho}{\rho}+O(x^{\frac{1}{3}})$

where ρ is a complex root of the Zeta-function, and

(7) $$\begin{aligned}\pi(x)&=\mathrm{Li}\,u-\frac{1}{\log u}\left(\sqrt{x}+\sum\frac{x^\rho}{\rho}\right)+O\left\{\frac{\sqrt{x}}{(\log u)^2}\right\}\\&=\mathrm{Li}\left\{x-\sqrt{x}-\sum\frac{x^\rho}{\rho}+O\left(\frac{\sqrt{u}}{\log u}\right)\right\}\\&=\mathrm{Li}\left\{\vartheta(x)+O\left(\frac{\sqrt{x}}{\log x}\right)\right\}\end{aligned}$$

where $\mathrm{Li}(x)=\int_0^x\frac{dt}{\log t}$.

It follows from (7) that

(8) $\vartheta(x)=\mathrm{Li}^{-1}(u)+O\left\{\frac{u}{(\log u)^2}\right\}$

But in terms of p we know only that

(9) $\vartheta(x)=p+O\{\sqrt{p}(\log p)^2\}$.

Let us consider more precisely the error in (8). We have

(10) $$\begin{aligned}\pi(x)&=\mathrm{Li}(u)-\frac{1}{\log u}\left(\sqrt{x}+\sum\frac{x^\rho}{\rho}\right)-\frac{1}{(\log u)^2}\left(2\sqrt{x}+\sum\frac{x^\rho}{\rho^2}\right)+O\left\{\frac{\sqrt{x}}{(\log u)^3}\right\}\\&=\mathrm{Li}\left\{\vartheta(x)-\frac{1}{\log u}\left(2\sqrt{x}+\sum\frac{x^\rho}{\rho}\right)+O\left\{\frac{\sqrt{x}}{(\log x)^2}\right\}\right.\end{aligned}$$

But

(11) $$\left|\sum\frac{x^\rho}{\rho^2}\right|\le\sum\left|\frac{x^\rho}{\rho^2}\right|=\sqrt{x}\sum\frac{1}{\rho(1-\rho)}=(2+\gamma-\log 4\pi)\sqrt{x}.$$

where γ is Euler's constant. It follows that

$$(12)\quad \begin{cases} \lim \left\{ \vartheta(x) - \mathrm{Li}^{-1}(x) \right\} \sqrt{\log x} = 4 + \gamma - \log 4\pi \ (= 2.046\ldots) \\ \lim \left\{ \vartheta(x) - \mathrm{Li}^{-1}(x) \right\} \sqrt{\log x} = \log 4\pi - \gamma \ (= 1.954\ldots) \end{cases}$$

Thus we see that if the series of the form (1) E can be expressed in terms of $\vartheta(x)$, then they can immediately be expressed in terms of $\pi(x)$ with the help of (8).

3. It is easy to show that, if $\phi'(x)$ is continuous between 2 and x then

$$(13)\quad \phi(2) + \phi(3) + \phi(5) + \cdots + \phi(p) = \pi(x)\,\phi(x) - \int_2^x \phi'(t)\,\pi(t)\,dt.$$

$$(14)\quad \phi(2)\log 2 + \phi(3)\log 3 + \phi(5)\log 5 + \cdots + \phi(p)\log p = \vartheta(x)\,\phi(x) - \int_2^x \phi(t)\,\vartheta(t)\,dt.$$

But it is easily seen that

$$(15)\quad \phi(t)\,\vartheta(x) - \int \phi'(t)\,\vartheta(t)\,dt = \int \phi(x)\,dx - \{x - \vartheta(x)\}\phi(x) + \int \phi'(t)\{x - \vartheta(t)\}\,dx.$$

Again we have by Taylor's theorem

$$(16)\quad \int_{\vartheta(a)}^{\vartheta(b)} \phi(t)\,dt = \int_a^b \phi(x)\,dx - \{x - \vartheta(x)\}\phi(x) + \tfrac{1}{2}\{x - \vartheta(x)\}^2 \phi'\{x(1-\theta) + \theta\vartheta(x)\},$$

where $0 \le \theta \le 1$. It follows from (14) – (16) that

$$(17)\quad \phi(2)\log 2 + \phi(3)\log 3 + \phi(5)\log 5 + \cdots + \phi(p)\log p = C + \int^{\vartheta(x)} \phi(t)\,dt + \int \phi'(t)\{x - \vartheta(t)\}\,dt - \tfrac{1}{2}\{x - \vartheta(x)\}^2 \phi'\{x(1-\theta) + \theta\vartheta(x)\}$$

4. Now let us consider the two forms

$$\sum_{p \le x} \frac{\log p}{p^s - 1}, \qquad \prod_{p \le x} (1 - p^{-s})$$

First let us assume all the known results about the primes, viz.

$$(18)\quad \begin{cases} \vartheta(x) = x + O\left\{x\, e^{-a\sqrt{\log x}}\right\} \\ \pi(x) = \mathrm{Li}\, x + O\left\{x\, e^{-a\sqrt{\log x}}\right\} \end{cases}$$

We have from (14) $\displaystyle\sum_{p \le x} \frac{\log p}{p^s - 1} = \frac{\vartheta(x)}{x^s - 1} + C(s) - \int \vartheta(x)\, d\left(\frac{1}{x^s - 1}\right)$,

where $C(s)$ is a constant depending on s only.

In other words

$$(19)\quad \begin{aligned} \sum_{p \le x} \frac{\log p}{p^s - 1} &= C(s) + \frac{x}{x^s - 1} - \int x\, d\left(\frac{1}{x^s - 1}\right) + O\left(\frac{x\, e^{-a\sqrt{\log x}}}{x^s - 1}\right) + \int_2^x O\left(x\, e^{-a\sqrt{\log x}}\right) d\frac{1}{x^s - 1} \\ &= C(s) + \int \frac{dx}{x^s - 1} + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right) \\ &= C(s) + \frac{x^{1-s}}{1-s} + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right) \end{aligned}$$

If $s > 1$, then $C(s)$ is obviously $\dfrac{1}{1-s} - \dfrac{\zeta'(s)}{\zeta(s)}$; and so

$$(20)\quad \sum_{p \le x} \frac{\log p}{p^s - 1} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{x^{1-s}}{1-s} + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right).$$

If $s < 1$, then $\displaystyle\sum_{p \le x} \frac{\log p}{p^{1-s}} = \frac{x^{1-s}}{1-s} + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right)$ (21)

If $s = 1$, it is easy to see from (19) and (20) that $\displaystyle\sum_{p \le x} \frac{\log p}{p - 1} = \log x - \gamma + O\left(e^{-a\sqrt{\log x}}\right)$. (22)

Similarly from (13) we can show that, if $s < 1$, then

$$(23)\quad \prod_{p \le x} \frac{1}{1 - p^{-s}} = e^{\mathrm{Li}(x^{1-s})} + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right)$$

If $s > 1$, then $\displaystyle\prod_{p \le x} \frac{1}{1 - p^{-s}} = \zeta(s)\left[e + \mathrm{Li}\,(x^{1-s}) + O\left(x^{1-s} e^{-a\sqrt{\log x}}\right)\right]$

and if $s = 1$, then (25) $\displaystyle\prod_{p \le x} \frac{1}{1 - \frac{1}{p}} = e^{\gamma \log x} + O\left(e^{-a\sqrt{\log x}}\right)$, using

[$\phi'(x)$ is a monotonic and continuous function such that $\log|\phi'(x)| = O(x)$]

1. Consider the function

$$F(x) = \sum_{p \le n} \phi\{\vartheta(x)\}\log p - \frac{1}{i}\phi\{\vartheta(x)\}\log x - \int^{\vartheta(x)} \phi(z)\,dz + i\log x \int_0^\infty \frac{\phi(\vartheta(x)+iz\log x) - \phi(\vartheta(x)-iz\log x)}{e^{2\pi z}-1}\,dz$$

Now $F(x)$ is obviously a continuous function except when x is of the form p. Also $\lim_{\epsilon\to 0} F(p+\epsilon) = F(p)$.

Again $\lim_{\epsilon\to 0} F(p-\epsilon) = F(p) - \phi\{\vartheta(p)\}\log p - \frac{1}{i}\phi\{\vartheta(p)-\log p\} + \frac{1}{i}\phi\{\vartheta(p)\}\log p + \int_{\vartheta(p)-\log p}^{\vartheta(p)} \phi(z)\,dz + i\log p \int_0^\infty \frac{\phi(\vartheta(p)-\log p+iz\log p) - \phi(\vartheta(p)-\log p - iz\log p)}{e^{2\pi z}-1}\,dz - i\log p\int_0^\infty \frac{\phi(\vartheta(p)+iz\log p) - \phi(\vartheta(p)-iz\log p)}{e^{2\pi z}-1}\,dz.$

But it is well known that

(1) $\frac{1}{k}\int_{n-k}^{n} \phi(z)\,dz = \frac{1}{i}\phi(n) + \frac{1}{i}\phi(n-k) + i\int_0^\infty \frac{\phi(n+ikz) - \phi(n-ikz)}{e^{2\pi z}-1}\,dz - i\int_0^\infty \frac{\phi(n-k+ikz) - \phi(n-k-ikz)}{e^{2\pi z}-1}\,dz.$

Hence $\lim_{\epsilon\to 0} F(p-\epsilon) = F(p)$.

Thus we see that $F(x)$ is continuous without exception.

Now the desired function $\bar{F}(x)$ viz.

$$-\frac{1}{2\pi}\phi\{\vartheta(x)\} + \frac{1}{\pi}\int_0^\infty \frac{\phi\{\vartheta(x)+iz\log x\} - \phi\{\vartheta(x)-iz\log x\}}{e^{2\pi z}-1}\,dz - \frac{\log x}{\pi}\int_0^\infty \frac{\phi'\{\vartheta(x)+iz\log x\} + \phi'\{\vartheta(x)-iz\log x\}}{e^{2\pi z}-1}\,z\,dz$$

is finite and continuous except at the isolated point $x=p$.

Hence $F(x) = \int^x \bar{F}(x)\,dx$

That is to say

(2) $\sum_{p\le x} \phi\{\vartheta(x)\}\log p = \int^{\vartheta(x)} \phi(z)\,dz + \frac{1}{2}\phi\{\vartheta(x)\}\log x - \frac{1}{2}\int_1^x \phi(\vartheta(x))\frac{dx}{x} + R(x)$

where

$$R(x) = \log x \int_0^\infty \frac{\phi(\vartheta(x)+iz\log x) - \phi(\vartheta(x)-iz\log x)}{i(e^{2\pi z}-1)}\,dz - \int_1^x \frac{dy}{y}\int_0^\infty \frac{\phi(\vartheta(y)+iz\log y) - \phi(\vartheta(y)-iz\log y)}{i(e^{2\pi z}-1)}\,dz - \int_1^x \frac{\log y}{y}\,dy \int_0^\infty z\,\frac{\phi'(\vartheta(y)+iz\log y) + \phi'(\vartheta(y)-iz\log y)}{e^{2\pi z}-1}\,dz.$$

Since the two sides do vanish when $x=1$, no constant is required. But if the integrals are either divergent or meaningless near the beginning, then we have to adjust the constant after choosing suitable lower limits for the integrals.

2. It is easy to see that

(3) $R(x) = O\left[|\phi'\vartheta(x)|\,(\log x)^2\right]$

From this and (2) it follows that

$$(4)\qquad \sum_{p\le x}\phi\{\vartheta(p)\}\log p = C + \int^{\vartheta(x)+\frac{1}{2}\log x}\phi(z)\,dz - \tfrac{1}{2}\int\frac{1}{z}\,\phi\{\vartheta(z)\}\,dz + O\left[\left|\phi\{\vartheta(x)\}\right|(\log x)^{\frac{1}{2}}\right]$$

For example when $\phi(x)=x^s$ we have

$$(5)\qquad \sum_{p\le x}\{\vartheta(p)\}^s\log p = C + \frac{1}{s+1}\{\vartheta^{s+1}(x)-1\} + \tfrac{1}{2}\vartheta^s(x)\log x - \tfrac{1}{2}\int^x\frac{1}{z}\,\vartheta^s(z)\,dz + O(x^{s-\frac{1}{2}}(\log x)^{\frac{1}{2}})$$
$$= C + \frac{1}{s+1}\{\vartheta^{s+1}(x)-1\} + \tfrac{1}{2}\vartheta^s(x)\log x + O(x^{s-\frac{1}{2}}),$$

for all values of s except 0. Here we have assumed the prime number theorem. If we assume the Riemann hypothesis we can easily show that the result is

$$(6)\qquad C + \frac{1}{s+1}\{\vartheta(x)-1\} + \tfrac{1}{2}\vartheta^s(x)\log x - \frac{x^{s+\rho}-1}{2s} + O(x^{s-\frac{1}{2}})$$

for all values of s (C depends on s only).

3. Another very interesting case is when $\phi=e^{-sz}$.

From (2) we have

$$(7)\qquad \sum_{p\le x}e^{-s\vartheta(p)}\log p = \int_0^{\vartheta(x)}e^{-sz}\,dz + \tfrac{1}{2}e^{-s\vartheta(x)}\log x - \tfrac{1}{2}\int_1^x e^{-s\vartheta(u)}\frac{du}{u} + R(s),$$

where
$$R(s) = -2\log x\; e^{-s\vartheta(x)}\int_0^\infty\frac{2\sin(2z\log u)}{e^{2\pi z}-1}\,dz + 2\int_1^x\frac{1}{y}\,e^{-s\vartheta(y)}\int_0^\infty\frac{2\sin(2z\log y)}{e^{2\pi z}-1}\,dz$$
$$+ 2s\int_1^x\frac{1}{y}\log y\; e^{-s\vartheta(y)}\,dy\int_0^\infty\frac{2\cos(2z\log y)}{e^{2\pi z}-1}\,dz$$

It follows that

$$(8)\qquad \sum_{p\le x}e^{-s\vartheta(p)}\log p = \frac{1}{s} - \frac{\log x}{x^s-1}\,e^{-s\vartheta(x)} - \int_1^x\frac{1-z^2+z^2\log z}{z(z^2-1)^2}\,e^{-s\vartheta(z)}\,dz$$

This suggests a more general case, viz.

$$\sum_{p\le x}e^{-s\vartheta(p)}f(p) = \frac{f(2)}{2^s-1} - e^{-s\vartheta(2)}\left\{\frac{f(2)}{2^s-1}-\frac{f(1)}{2^s-1}\right\} + e^{-s\vartheta(3)}\left\{\frac{f(3)}{3^s-1}-\frac{f(2)}{5^s-1}\right\}$$
$$+ e^{-s\vartheta(5)}\left\{\frac{f(5)}{7^s-1}-\frac{f(3)}{5^s-1}\right\} + \cdots + e^{-s\vartheta(p')}\left\{\frac{f(p)}{p^s-1}-\frac{f(p')}{p'^s-1}\right\}$$
$$+ e^{-s\vartheta(x)}\left\{\frac{f(x)}{x^s-1}-\frac{f(p)}{p^s-1}\right\} \equiv -\frac{f(x)}{x^s-1}e^{-s\vartheta(x)},$$

where p' is the prime next below p. In other words

$$(9)\qquad \sum_{p\le x}e^{-s\vartheta(p)}f(p) = \frac{f(2)}{2^s-1} - \frac{f(x)}{x^s-1}\,e^{-s\vartheta(x)} + \int_2^x e^{-s\vartheta(z)}\left(\frac{d}{dz}\,\frac{f(z)}{z^s-1}\right)dz$$
$$= \frac{f(2)}{2^s-1} - \frac{f(x)}{x^s-1}\,e^{-s\vartheta(x)} + \int_2^x e^{-s\vartheta(z)}\left\{\frac{f'(z)}{z^s-1} - \frac{sz^{s-1}f(z)}{(z^s-1)^2}\right\}dz.$$

If we suppose that $f(x)=1-(9)$, then

$$(10)\qquad \sum_{p\le x}e^{-s\vartheta(p)} = \frac{1}{2^s-1} - \frac{e^{-s\vartheta(x)}}{x^s-1} - s\int_2^x\frac{z^{s-1}e^{-s\vartheta(z)}}{(z^s-1)^2}\,dz.$$

4. Making x tend to ∞ in (4) and (11) we obtain

$$(11)\qquad \phi\{\vartheta(2)\}\log 2 + \phi\{\vartheta(3)\}\log 3 + \phi\{\vartheta(5)\}\log 5 + \phi\{\vartheta(7)\}\log 7 + \cdots$$
$$= \int_0^\infty\phi(u)\,du - \tfrac{1}{2}\int_1^\infty\frac{1}{u}\,\phi(\vartheta(u))\,du - \int_0^\infty\frac{2}{\pi}\int_0^\infty\frac{1}{u}\,\phi(\vartheta(u)+z\log u)\,du\;\frac{\sin(2z\log u)}{e^{2\pi z}-1}\,dz$$
$$= \int_0^\infty\phi(u)\,du - \tfrac{1}{2}\int_1^\infty\frac{1}{u}\,\phi(\vartheta(u))\,du - \int_0^\infty\frac{2}{\pi}\,dz\int_1^\infty\phi(\vartheta(u)+z\log u)\,\frac{\sin(2z\log u)}{e^{2\pi z}-1}\,\frac{du}{u}$$

(12) $2^{-s}f(2)+6^{-s}f(3)+30^{-s}f(5)+210^{-s}f(7)+\cdots$
$= \frac{f(2)}{2^{s}-1} + \int_1^\infty e^{-s\vartheta(x)}\left\{\frac{f(x)}{x^s-1} - \frac{s\,x^{s-1}f(x)}{(x^s-1)^2}\right\}dx.$

As particular cases of (12) we have

(13) $2^{-s}+6^{-s}+30^{-s}+210^{-s}+\cdots = \frac{1}{2^s-1} - s\int_1^\infty \frac{x^{s-1}e^{-s\vartheta(x)}}{(x^s-1)^2}dx$

(14) $2^{-s}\log 2 + 6^{-s}\log 3 + 30^{-s}\log 5 + \cdots = \frac{1}{s} - \int_1^\infty e^{-s\vartheta(x)}\frac{1-x^s+sx^s\log x}{x(1-x^s)^2}dx.$

5. Let us consider the behaviour of (14) and (15) as $s\to 0$. Now

(15) $2^{-s}\log 2 + 6^{-s}\log 3 + 30^{-s}\log 5 + \cdots = \frac{1}{s} - \int_1^\infty e^{-s\vartheta(x)}\left\{\frac{1}{2} - \frac{s\log x}{6} + \frac{s^2(\log x)^2}{180} - \cdots\right\}\frac{dx}{x}$

$= \frac{1}{s} - \frac{1}{2}\int_1^\infty e^{-s\vartheta(x)}\frac{dx}{x} + O\left\{s(\log s)^2\right\}$

$= \frac{1}{s} + \frac{1}{2}\log s + C + O(\sqrt{s})$ (Riemann's hypothesis)

$= \frac{1}{s} + \frac{1}{2}\log s + C + O\left\{e^{-a\sqrt{\log(1/s)}}\right\}$ (Prime number theorem)

$= \frac{1}{s} + \frac{1}{2}\log s + O(1)$ (Elementary reasoning)

Again

(16) $2^{-s}+6^{-s}+30^{-s}+210^{-s}+\cdots = \frac{1}{2^s-1} - \int_2^\infty e^{-s\vartheta(x)}\left\{\frac{1}{s(\log x)^2} - \frac{s}{12} + \frac{s^2(\log x)^2}{240} - \cdots\right\}\frac{dx}{x}$

$= \frac{1}{2^s-1} - \frac{1}{s}\int_2^\infty \frac{e^{-s\vartheta(x)}}{x(\log x)^2}dx + O(s\log s)$

$= \frac{1}{2^s-1} - \frac{1}{s}\int_1^\infty \frac{e^{-sx}}{x(\log x)^2}dx + O\left(\frac{e^{-a\sqrt{\log 1/s}}}{s}\right)$

$= \frac{1}{2^s-1} - \frac{e^{-2s}}{s\log 2} + \int_2^\infty \frac{e^{-sx}}{\log x}dx + O\left(\frac{e^{-a\sqrt{\log 1/s}}}{s}\right)$

$= -\frac{1}{s\log s} + \frac{\gamma_0}{s(\log s)^2} - \frac{\gamma_1}{s(\log s)^3} + \frac{\gamma_2}{s(\log s)^4} - \cdots + O\left\{\frac{1}{s(\log\frac{1}{s})^n}\right\};$

where $\Gamma(1+z) = 1 - \frac{\gamma_0 z}{1!} + \frac{\gamma_1 z^2}{1!} - \frac{\gamma_2 z^3}{3!} - \cdots$

so that γ_0 is Euler's constant. Without assuming the prime number theorem we can prove that (16) is of the form

$$-\frac{1}{s\log s} + O\left\{\frac{1}{s(\log s)^2}\right\}.$$

Hence

(17) $(1-2^{-s})(1-6^{-s})(1-30^{-s})(1-210^{-s})\cdots = e^{\frac{\pi^2}{6s\log\frac{1}{s}} + O\left\{\frac{1}{s(\log s)^2}\right\}}.$

I. Suppose that A and B are any two integers such that $A^2+3B^2=p$ and $A\equiv 1 \pmod 3$ p being a prime of the form $6k+1$.

Let $\sum_1^\infty \Omega_0(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4$

$\sum_1^\infty \Omega_1(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4\{1-24(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\cdots)\}$

$\sum_1^\infty \Omega_2(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4\{1+240(\frac{x^3}{1-x^3}+\frac{2x^6}{1-x^6}\cdots)\}$

$\sum_1^\infty \Omega_3(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4\{1-504(\frac{x^5}{1-x^5}+\cdots)\}$.

$\left\{\begin{array}{l}\sum_1^\infty \Omega_4'(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^8\{1+480(\frac{x^4}{1-x^4}+\cdots)\}\\ \sum_1^\infty \Omega_4''(n)\, x^n = x^{\frac{5}{3}}\{(1-x)(1-x^2)\cdots\}^{20}\end{array}\right.$

$\Omega_4(n) = \Omega_4'(n) + 288\,\omega\sqrt{70}\,\Omega_4''(n)$, $\omega^3=1$.

$\sum_1^\infty \Omega_5(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4\{1-264(\frac{x^4}{1-x^4}+\cdots)\}$

$\left\{\begin{array}{l}\sum_1^\infty \Omega_7'(n)\, x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^4\{1-24(\frac{x^2}{1-x^2}+\cdots)\}\\ \sum_1^\infty \Omega_7''(n)\, x^n = x^{\frac{5}{3}}\{(1-x)(1-x^2)\cdots\}^{10}\{1-504(\frac{x^4}{1-x^4}+\cdots)\}\end{array}\right.$

$\Omega_7(n) = \Omega_7'(n) + 10080\,\omega\sqrt{286}\,\Omega_7''(n)$ $\omega^3=1$.

In all these cases

$$\sum_1^\infty \frac{\Omega_\lambda(n)}{n^s} = \prod_p \frac{1}{1-\Omega_\lambda(p)\,p^{-s}+p^{2\lambda+1-2s}}$$

where p assumes all prime values greater than 3.

If $\lambda = 0, 2, 3, 5$ then

$\Omega_\lambda(p)=0$ $p\equiv -1 \pmod 6$, $\Omega_\lambda(p) = (A+iB\sqrt{3})^{2\lambda+1}+(A-iB\sqrt{3})^{2\lambda+1}$ $p\equiv 1 \pmod 6$,

$\Omega_\lambda(n) = n\,\Omega_0(n)$ for all values of n. But $\Omega_4(p)$ and $\Omega_7(p)$ do not seem to have such simple laws.

II. Suppose again that A and B are defined as in I and let

$\sum_1^\infty \Omega_0(n)\, x^{\frac{n}{3}} = x^{\frac{1}{3}}\{(1-x)(1-x^2)(1-x^3)\cdots\}^8$

$\sum_1^\infty \Omega_1(n)\, x^{\frac{n}{3}} = x^{\frac{1}{3}}\{(1-x)(1-x^2)(1-x^3)\cdots\}^8\{1-24(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\cdots)\}$

$\left\{\begin{array}{l}\sum_1^\infty \Omega_2'(n)\, x^{\frac{n}{3}} = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^8\{1+240(\frac{x^3}{1-x^3}+\cdots)\}\\ \sum_1^\infty \Omega_2''(n)\, x^{\frac{n}{3}} = x^{\frac{5}{3}}\{(1-x)(1-x^2)\cdots\}^{16}\end{array}\right.$

$\Omega_2(n) = \Omega_2'(n) + 6\,\omega\sqrt{10}\,\Omega_2''(n)$, $\omega^3=1$.

$\sum_1^\infty \Omega_3(n)\, x^{\frac{n}{3}} = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^8\{1-504(\frac{x^3}{1-x^3}+\cdots)\}$

$\left\{\begin{array}{l}\sum_1^\infty \Omega_4'(n)\, x^{\frac{n}{3}} = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^8\{1+480(\frac{x^3}{1-x^3}+\cdots)\}\\ \sum_1^\infty \Omega_4''(n)\, x^{\frac{n}{3}} = x^{\frac{5}{3}}\{(1-x)(1-x^2)\cdots\}^{16}\{1+240(\frac{x^3}{1-x^3}+\cdots)\}\end{array}\right.$

$\Omega_4(n) = \Omega_4'(n) + 6\,\omega\sqrt{70}\,\Omega_4''(n)$, $\omega^3=1$

$$\begin{cases} \sum_1^\infty \Omega_3'(n)x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^{12}\{1-264\left(\frac{x^n}{1-x^n}+\cdots\right)\} \\ \sum_1^\infty \Omega_3''(n)x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^{12}\{1-504\left(\frac{x^n}{1-x^n}+\cdots\right)\} \end{cases}$$

$$\Omega_5(n) = \Omega_5'(n) + 12\omega\sqrt{33}\,\Omega_5''(n); \quad \omega^2 = 1$$

$$\begin{cases} \sum_1^\infty \Omega_7'(n)x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^8\{1-24\left(\frac{x^n}{1-x^n}+\cdots\right)\} \\ \sum_1^\infty \Omega_7''(n)x^n = x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}^{16}\{1-264\left(\frac{x^n}{1-x^n}+\cdots\right)\} \end{cases}$$

$$\Omega_7(n) = \Omega_7'(n) + 12\omega\sqrt{910}\,\Omega_7''(n); \quad \omega^2 = 1$$

In all these cases

$$\sum_1^\infty \frac{\Omega_\lambda(n)}{n^s} = \frac{1}{1-\Omega_\lambda(p)p^{-s}+p^{\lambda+2-2s}} \qquad \Omega_1(n) = n\Omega_0(n)$$

where p assumes all prime values except 3.

If $\lambda = 0$ or 3, then $\begin{cases} \Omega_\lambda(p) = 0 & p \equiv -1 \pmod 3 \\ \Omega_\lambda(p) = (A+iB\sqrt{3})^{\lambda+2} + (A-iB\sqrt{3})^{\lambda+2} & p \equiv 1 \pmod 6 \end{cases}$ $\Omega_\lambda(n) = \Omega_0(n)$ [illegible]

III. Suppose that A and B are integers such that $A^2+4B^2 = p$ where p is of the form $4k+1$.

$$\sum_1^\infty \Omega_0(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6$$

$$\sum_1^\infty \Omega_1(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\cdots\right)\}$$

$$\sum_1^\infty \Omega_2(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1+240\left(\frac{x^n}{1-x^n}+\frac{2^3x^{2n}}{1-x^{2n}}+\cdots\right)\}$$

$$\begin{cases} \sum_1^\infty \Omega_3'(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1-504\left(\frac{x^n}{1-x^n}+\frac{2^5x^{2n}}{1-x^{2n}}+\cdots\right)\} \\ \sum_1^\infty \Omega_3''(n)x^n = x^{\frac{3}{12}}\{(1-x)(1-x^2)\cdots\}^{18} \end{cases}$$

$$\Omega_3(n) = \Omega_3'(n) + 24\omega\sqrt{35}\,\Omega_3''(n); \quad \omega^2 = -1.$$

$$\sum_1^\infty \Omega_4(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1+480\left(\frac{x^n}{1-x^n}+\cdots\right)\}$$

$$\begin{cases} \sum_1^\infty \Omega_5'(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1-264\left(\frac{x^n}{1-x^n}+\cdots\right)\} \\ \sum_1^\infty \Omega_5''(n)x^n = x^{\frac{3}{12}}\{(1-x)(1-x^2)\cdots\}^{18}\{1+240\left(\frac{x^n}{1-x^n}+\cdots\right)\} \end{cases}$$

$$\Omega_5(n) = \Omega_5'(n) + 24\omega\sqrt{1155}\,\Omega_5''(n); \quad \omega^2 = -1.$$

$$\begin{cases} \sum_1^\infty \Omega_7'(n)x^n = x^{\frac{1}{12}}\{(1-x)(1-x^2)\cdots\}^6\{1-24\left(\frac{x^n}{1-x^n}+\cdots\right)\} \\ \sum_1^\infty \Omega_7''(n)x^n = x^{\frac{3}{12}}\{(1-x)(1-x^2)\cdots\}^{18}\{1+480\left(\frac{x^n}{1-x^n}+\cdots\right)\} \end{cases}$$

$$\Omega_7(n) = \Omega_7'(n) + 120\omega\sqrt{3005}\,\Omega_7''(n) \qquad \omega^2 = -1$$

In all these cases

$$\sum_1^\infty \frac{\Omega_\lambda(n)}{n^s} = \Pi_1\,\Pi_2$$

where $\Pi_1 = \prod_p \frac{1}{1-\Omega_\lambda(p)p^{-s}-p^{\lambda+2-2s}}$

p assuming ~~all values~~ primes of the form $4k-1$ and

$$\Pi_2 = \prod_p \frac{1}{1-\Omega_\lambda(p)p^{-s}+p^{\lambda+2-2s}}$$

p assuming all primes of the form $4k+1$

If $\lambda = 0, 2$ or 4, then

$$\begin{cases} \Omega_\lambda(p) = 0 & p \equiv -1 \pmod 4 \\ \Omega_\lambda(p) = (A+B2i)^{\lambda+2} + (A-2iB)^{\lambda+2} & p \equiv 1 \pmod 4 \end{cases}$$

$$\Omega_1(n) = n\,\Omega_0(n)$$

(IV) Let

$$\sum_1^\infty \Omega_0(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^2$$

$$\sum_1^\infty \Omega_1(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^2\{1-24(\tfrac{x}{1-x}+\cdots)\}$$

$$\begin{cases} \sum_1^\infty \Omega_1'(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^4\{1+240(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_2''(n)\, x^{\frac{n}{2}} = x^{\frac{5}{2}}\{(1-x)(1-x^2)\cdots\}^{10} \end{cases}$$

$$\Omega_2(n) = \Omega_1'(n) + 48\,\omega\,\Omega_2''(n);\quad \omega^2 = +1$$

$$\begin{cases} \sum_1^\infty \Omega_3'(n)\, x^{\frac{n}{2}} = x^{\frac{3}{2}}\{(1-x)(1-x^2)\cdots\}^6\{1-504(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_3''(n)\, x^{\frac{n}{2}} = x^{\frac{7}{2}}\{(1-x)(1-x^2)\cdots\}^{14} \end{cases}$$

$$\Omega_3(n) = \Omega_3'(n) + 360\,\omega\,\omega_1^3\,\Omega_3''(n);\quad \omega^2 = -1.$$

$$\begin{cases} \sum_1^\infty \Omega_4'(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^8\{1+480(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_4''(n)\, x^{\frac{n}{2}} = x^{\frac{5}{2}}\{(1-x)(1-x^2)\cdots\}^{20}\{1+240(\tfrac{x}{1-x}+\cdots)\} \end{cases}$$

$$\Omega_4(n) = \Omega_4'(n) + 672\,\omega\,\Omega_4''(n);\quad \omega^2 = 1.$$

$$\begin{cases} \sum_1^\infty \Omega_5'(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^4\{1-264(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_5''(n)\, x^{\frac{n}{2}} = x^{\frac{5}{2}}\{(1-x)(1-x^2)\cdots\}^{10}\{1-504(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_5'''(n)\, x^{\frac{n}{2}} = x^{\frac{9}{2}}\{(1-x)(1-x^2)\cdots\}^{16}\{1+240(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_5''''(n)\, x^{\frac{n}{2}} = x^{\frac{13}{2}}\{(1-x)(1-x^2)\cdots\}^{22} \end{cases}$$

$$\Omega_5(n) = \Omega_5'(n) + 96\,\omega_1\sqrt{1045}\,\Omega_5''(n) + 216\,\omega_2\sqrt{7315}\,\Omega_5'''(n) + 103680\,\omega_1\omega_2\sqrt{7}\,\Omega_5''''(n).$$

$$\omega_1^2 = 1,\ \omega_2^2 = -1.$$

$$\begin{cases} \sum_1^\infty \Omega_7'(n)\, x^{\frac{n}{2}} = x^{\frac{1}{2}}\{(1-x)(1-x^2)\cdots\}^2\{1-24(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_7''(n)\, x^{\frac{n}{2}} = x^{\frac{5}{2}}\{(1-x)(1-x^2)\cdots\}^{10}\{1-264(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_7'''(n)\, x^{\frac{n}{2}} = x^{\frac{9}{2}}\{(1-x)(1-x^2)\cdots\}^{18}\{1+480(\tfrac{x}{1-x}+\cdots)\} \\ \sum_1^\infty \Omega_7''''(n)\, x^{\frac{n}{2}} = x^{\frac{13}{2}}\{(1-x)(1-x^2)\cdots\}^{26}\{1+240(\tfrac{x}{1-x}+\cdots)\} \end{cases}$$

$$\Omega_7(n) = \Omega_7'(n) + 48\,\omega_1\sqrt{910.2911}\,\Omega_7''(n) + 216\,\omega_2\sqrt{(5005.2911)}\,\Omega_7'''(n) - 471744\,\omega_1\omega_2\sqrt{22}\,\Omega_7''''(n)$$

$$\omega_1^2 = 1,\ \omega_2^2 = -1$$

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$$G(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \frac{x}{1-x} + \frac{x^2}{(1-x)(1-x^2)} + \cdots$$

$$H(x) = \frac{1}{(1-x^2)(1-x^4)(1-x^6)\cdots} = 1 + \frac{x^2}{1-x^2} + \frac{x^4}{(1-x^2)(1-x^4)} + \cdots$$

$$\chi(x) = (1+x)(1+x^3)(1+x^5)\cdots$$

$$\{G(x)\}^4 H(x) - x^2 G(x)\{H(x)\}^4 = 1 + 4x\{G(x)H(x)\}^2$$

$$G(x)G(x^4) + xH(x)H(x^4) = \{G(x^2)\}^2 = \frac{1+2x+2x^4+2x^9+\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots}$$

$$G(x)G(x^4) - xH(x)H(x^4) = \frac{1+2x^2+2x^8+2x^{18}+\cdots}{(1-x^4)(1-x^8)(1-x^{12})\cdots}$$

$$G(x^4)H(x) - x^3 G(x)H(x^4) = 1$$

$$G(x^4)H(x) - x^3 G(x)H(x^4) = \chi(x^4)$$

$$G(x)G(x^8) + x^3 H(x)H(x^8) = \frac{(1-x^{16})(1-x^{24})(1-x^{40})(1-x^{48})\cdots}{(1-x^8)(1-x^{16})(1-x^{24})(1-x^{32})\cdots}$$

$$\begin{cases} G(x^3)G(x^5) + xH(x^3)H(x^5) = \dfrac{\chi(-x^3)}{\chi(-x)} \\ G(x^9)H(x) - xG(x)H(x^9) = \dfrac{\chi(-x^9)}{\chi(-x^3)} \end{cases}$$

$$\begin{cases} G(x^3)H(x^5) - xG(x^5)H(x^3) = \dfrac{\chi(-x)}{\chi(-x^3)} \\ G(x)G(x^{15}) + x^3 H(x)H(x^{15}) = \dfrac{\chi(-x^5)}{\chi(-x)} \end{cases}$$

$$\begin{cases} G(x^9)H(x^3) - xG(x^3)H(x^9) = \dfrac{\chi(-x)\chi(-x^9)}{\chi(-x^3)\chi(-x^{27})} \\ G(x)G(x^{27}) + x^9 H(x)H(x^{27}) = \dfrac{\chi(-x^3)\chi(-x^{27})}{\chi(-x)\chi(-x^9)} \end{cases}$$

$$\begin{cases} G(x^9)H(x^7) - xG(x^7)H(x^9) = \dfrac{\chi(-x^7)\chi(-x^9)}{\chi(-x^{63})\chi(-x^{81})} \\ G(x^{25})H(x) - x^{24}G(x)H(x^{25}) = \dfrac{\chi(-x^5)\chi(-x^{25})}{\chi(-x)\chi(-x^{125})} \end{cases}$$

$$G(x^3)G(x^7) + x^2 H(x^3)H(x^7) = G(x^{11})H(x) - x^5 G(x)H(x^{11})$$
$$= \frac{1}{2\pi i}\chi(x^{\frac{1}{2}})\chi(-x^{\frac{3}{2}})\chi(x^{\frac{5}{2}})\chi(-x^{\frac{11}{2}}) - \frac{1}{2\pi i}\chi(-x^{\frac{1}{2}})\chi(x^{\frac{3}{2}})\chi(-x^{\frac{5}{2}})\chi(x^{\frac{11}{2}})$$

$$G(x^5)G(x^{11}) + x^3 H(x^5)H(x^{11}) = G(x^{21})H(x) - x^5 G(x)H(x^{21})$$
$$= \sqrt{\left\{\frac{\chi(-x^{11})}{\chi(-x)} - x\frac{\chi(-x)}{\chi(-x^{11})}\right\}}$$

$$G(x)G(x^{11}) + x^5 H(x)H(x^{11}) = \frac{1}{4x^{\frac{5}{2}}}\left\{\chi(x^{\frac{1}{2}})\chi(x^{\frac{11}{2}})\right\}^2 - \frac{1}{4x^{\frac{5}{2}}}\left\{\chi(-x^{\frac{1}{2}})\chi(-x^{\frac{11}{2}})\right\}^2$$

$$G(x^{21})H(x) - x^{20}G(x)H(x^{21}) = \frac{1}{2x^{10}}\chi(x)\chi(x^{21}) - \frac{1}{2x^{10}}\chi(-x)\chi(-x^{21}) + \frac{x^3}{\chi(-x^7)\chi(-x^{21})}$$

$$G(x)G(x^{27}) + x^8 H(x)H(x^{27}) = \{(1-x)(1-x^3)(1-x^5)\cdots\}\{(1-x^{27})(1-x^{81})\cdots\}$$
$$= G(x^{13})H(x^{13}) - x^{13}G(x^{13})H(x^{13})\{(1-x)(1-x^3)(1-x^5)\cdots\}\{(1-x^{13})(1-x^{39})\cdots\}$$
$$- \frac{1}{2x^{13}}\{(1-2x^{13}+2x^{52}-\cdots)\} - \frac{1}{2x^{13}}\{(1-2x^{13}+2x^{52}-\cdots)\}$$

$$\begin{cases} G(x)\,H(-x)+G(-x)\,H(x)=\dfrac{2}{\chi^2(-x^2)} \\ G(x)\,H(-x)-G(-x)\,H(x)=\dfrac{2x}{(1-x^2)(1-x^3)(1-x^4)\cdots} \end{cases}$$

$$G(x)/G(-x)+x\,H(x)/H(-x)=\chi(x)$$

$$G(x^2)/G(-x^2)+x\,H(x^2)/H(-x^2)=\frac{\chi(x)\chi(x^2)}{\chi(x^4)\chi(x^3)}$$

$$G(x^2)\,H(x^3)-x\,H(x^2)\,G(x^3)=\frac{\chi(x)\chi(x^2)}{\chi(x^6)}$$

$$G(x)\,G(x^3)-x^2\,H(x)\,H(x^3)=\frac{\chi(x)\chi(x^4)}{\chi(x^2)}$$

$$G(x^2)\,H(x^4)+x\,G(x)\,H(x^3)=\frac{\chi(x)\chi(x^3)}{\chi(x^2)\chi(x^6)}\cdot\frac{2x^3}{\chi(-x^2)\chi(-x^3)\chi(-x^4)\chi(-x^6)}$$

$$G(x)\,G(x^4)+x^4\,H(x)\,H(x^4)=U$$

$$G(x^3)\,G(x^4)+x^3\,H(x^3)\,H(x^4)=V$$

$$\text{Here } U^2+xV^2=\chi^3(x)\chi^3(x^4) \qquad UV+x=\chi^3(x)\chi^3(x^4)$$

$$G(x^2)\,H(x^3)-x^2\,G(x^3)\,H(x^4)=U$$

$$G(x)\,H(x^2)+x^3\,G(x^3)\,H(x^4)=V$$

$$\text{Here } \frac{U}{V}=\frac{\chi(-x)}{\chi(-x^7)}\ ;\quad (UV)^2-x=(UV)^2=\left\{\frac{\chi(-x^7)}{\chi(-x)}\right\}^2\left[1+x^3\left\{\frac{\chi(x)}{\chi(-x)}\right\}^8\right]^2$$

$$\left\{G(x^3)\,G(x^{13})+x^5\,H(x^3)\,H(x^{13})\right\}\left\{G(x^9)\,H(x^{13})-x\,G(x^{13})\,H(x^9)\right\}$$
$$=\chi(-x)\chi(-x^{13})+x+\frac{2x^2}{\chi(-x)\chi(-x^{13})\chi(-x^{26})}$$

$$\frac{G(x^2)\,G(x^{14})-x^3\,G(x^7)\,H(x^{14})}{G(x^{28})\,H(x)+x^5\,G(x)\,H(x^{28})}=\frac{\chi(-x^4)}{\chi(-x^{28})}$$

$$\frac{G(x^3)\,G(x^{23})+x^2\,H(x^3)\,H(x^{23})}{G(x)\,H(x^{69})-x\,H(x^{23})\,G(x^{69})}=\frac{\chi(-x^2)}{\chi(-x^{23})}$$

$$\frac{G(x^3)\,G(x^{22})+x^5\,H(x^3)\,H(x^{11})}{G(x^2)\,H(x^{33})-x\,G(x^{11})\,H(x^{22})}=\frac{\chi(-x^{11})}{\chi(-x)}$$

$$\frac{G(x)\,G(x^{43})+x^5\,H(x)\,H(x^{43})}{G(x^{11})\,H(x^{43})-x^5\,G(x^{43})\,H(x^{11})}=\frac{\chi(-x^{11})\chi(-x^2)}{\chi(-x)\chi(-x^{43})}$$

$$\left\{G(x)\,G(x^{11})+x^4\,H(x)\,H(x^{11})\right\}\left\{G(x^{11})\,H(x^{11})-x^4\,G(x)\,H(x)\right\}$$
$$=G(x^{11})\,G(x^{22})+x^8\,H(x^{11})\,H(x^{22})$$

$$\left\{G(x)\,G(x^{11})+x^4\,H(x)\,H(x^{11})\right\}\left\{G(x^{11})\,H(x)+x^4\,G(x)\,H(x^{11})\right\}$$
$$=\chi(-x)\chi(-x^{11})+2x^2+\frac{4x^3}{\chi(-x)\chi(-x^{11})}+\sqrt{\left\{\chi(-x)\chi(-x^{11})+\frac{2x^3}{\chi(-x)\chi(-x^{11})}\right\}^2+4x^6}$$

[Middle of a paper?]

II (35)

Modul: 5 and 25

20. In this second part we shall use J_1, J_2, J_3 and G_1, G_2, G_3 to denote functions of x with integral powers of x as well as integral coefficients. These are the same functions in the same section unlike J. We shall also use J in the same sense as in the first part.

We start with Euler's identity

$$(20.1)\quad (1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - \cdots$$

and Jacobi's identity

$$(20.11)\quad \{(1-x)(1-x^2)(1-x^3)\cdots\}^3 = 1 - 3x + 5x^2 - 7x^3 + \cdots$$

It is easy to see from 20.1 that

$$(20.2)\quad \frac{(1-x^{\frac{1}{2}})(1-x^{\frac{3}{2}})(1-x^{\frac{5}{2}})\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots} = J_1 - x^{\frac{1}{2}} + x^{\frac{1}{2}} J_2$$

Now cubing both sides we obtain

$$(20.3)\quad \frac{1 - 3x^{\frac{1}{2}} + 5x^{\frac{2}{2}} - 7x^{\frac{3}{2}} + \cdots}{1 - 3x^2 + 5x^4 - 7x^6 + \cdots} = (J_1^3 - 2J_1^2 x) - x^{\frac{1}{2}}(3J_1^2 - J_1^3 x) + 3J_1 x^{\frac{1}{2}}(1 + J_1 J_2) - x^{\frac{3}{2}}(1 + 6J_1 J_2) + 3J_2 x^{\frac{1}{2}}(1 + J_1 J_2)$$

But it is easy to see that

$$(20.31)\quad \frac{1 - 3x^{\frac{1}{2}} + 5x^{\frac{2}{2}} - 7x^{\frac{3}{2}} + \cdots}{1 - 3x^2 + 5x^4 - 7x^6 + \cdots} = G_1 + x^{\frac{1}{2}} G_2 + 5x^{\frac{3}{2}}$$

Hence

$$(20.4)\quad J_1(1 + J_1 J_2) = 0,\quad 1 + 6J_1 J_2 = -5,\quad J_2(1 + J_1 J_2) = 0$$

These three equations give one and the same relation between J_1 and J_2 viz

$$J_1 J_2 = -1. \qquad (36)$$

Using this we obtain

$$(20.5)\quad \frac{(1-x^2)(1-x^{10})(1-x^{12})\cdots}{(1-x^{\frac{1}{2}})(1-x^{\frac{5}{2}})(1-x^{\frac{7}{2}})\cdots} = \frac{1}{J_1 - x^{\frac{1}{2}} + x^{\frac{1}{2}} J_2} = \frac{(J_2^4 + 3J_2^2 x) + x^{\frac{1}{2}}(J_1^3 + 2J_2^2 x) + x^{\frac{2}{2}}(2J_1^2 + J_2^3 x) + x^{\frac{3}{2}}(3J_1 + J_2^4 x) + 5x^{\frac{4}{2}}}{J_1^5 - 11x + x^2 J_2^5}$$

by rationalising the denominator $J_1 - x^{\frac{1}{2}} + x^{\frac{1}{2}} J_2$

It follows from (20.5) that

$$(20.6)\quad \{p(4) + p(9)x + p(14)x^2 + p(19)x^3 + \cdots\}\{(1-x^2)(1-x^4)(1-x^6)\cdots\} = \frac{5}{J_1^5 - 11x + x^2 J_2^5}$$

But we see from (20.2) that

$$(20.21)\quad \frac{(1-\omega x^{\frac{1}{2}})(1-\omega^2 x^{\frac{3}{2}})(1-\omega^3 x^{\frac{5}{2}})\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots} = J_1 - \omega x^{\frac{1}{2}} + \omega^2 x^{\frac{2}{2}} J_2$$

where $\omega^5 = 1$. Now writing the five values of ω in (20.21) and multiplying them together we obtain.

(20.7) $\left\{\frac{(1-x)(1-x^5)(1-x^{10})\cdots}{(1-x^2)(1-x^{10})(1-x^{15})\cdots}\right\}^6 = J_1^5 - 11x + x^2 J_2^5.$

It follows from this and 20.6 that

(20.8) $p(4) + p(9)x + p(14)x^2 + p(19)x^3 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6}$

It follows that

(20.81) $p(5n-1) \equiv 0 \pmod 5$ (37)

Again the right hand side in ( ) is of the form

$5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})^{2\cdots}\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^5} + 25J$

It follows from this and (20.81) that the coefficients of $x^5, x^{10}, x^{15}, \cdots$ in this are all multiples of 25 and consequently the coefficient of x^{5n-1} in the left hand side of (20.8) is a multiple of 25. In other words

(20.82) $p(25n-1) \equiv 0 \pmod{25}$

It follows also from (20.8) that

(20.9) $p(4) + p(9)x + p(14)x^2 + p(19)x^3 + \cdots = 5\{(1-x)(1-x^2)(1-x^3)\cdots\}^{19} + 125J.$

Modulus 125

21. Changing x to x^5 in (20.8) and arguing as before, using (20.5) and (20.7) we find that

$$\begin{aligned}
(21.1)\quad & p(24) + p(49)x + p(74)x^2 + p(99)x^3 + \cdots \\
&= 5^2\cdot 67\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^6}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^7} \\
&\quad + 5^5\cdot 52x\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{12}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{13}} \\
&\quad + 5^7\cdot 63x^2\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{18}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{19}} \\
&\quad + 5^{10}\cdot 6x^3\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{24}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{25}} \\
&\quad + 5^{12}x^4\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{30}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{31}}.
\end{aligned}$$

Now

(21.2) $\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^6}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^7} = (1-7x+5x^2-7x^6+\cdots)\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^6 + 5J$ (38)

and the coefficients of $x^{5n-1}, x^{5n-2}, x^{5n-3}$ in $1-7x+5x^2-7x^6+\cdots$ is easily seen to be zero or multiples of 5. It follows that the coefficients of $x^{5n-1}, x^{5n-2}, x^{5n-3}$ in the left hand side of (21.1) are multiples of 125. In other words

(21.3) $\begin{cases} p(125n-1) \\ p(125n-26) \\ p(125n-51) \end{cases} \equiv 0 \pmod{125}$

It is also easy to see from (21.1) that

(21.4) $p(24) + p(124)x + p(224)x^2 + p(324)x^3 + \cdots = 75\cdot\{(1-x)(1-x^2)(1-x^3)\cdots\}^{23} + 125J.$

The right hand side in ( ) can be written in the form

(21.5) $75\frac{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}}{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{23}} + 125J$

But it is easy to shew that

(21.6) $\qquad (Q^3-R^2)^3 = 2\sum_1^\infty (n^3-n)\,\sigma_3(n)\,x^n + 5^3 J$

It follows that

(21.7) $\qquad \{p(24)x^3 + p(49)x^7 + p(74)x^{10} + \cdots\}(1-x^{24})(1-x^{49})\cdots$
$\qquad = 25\sum_1^\infty (n^3-n)\,\sigma_3(n)\,x^n + 125\,J.$

In the words

(21.8) $\quad p(25n-24) - p(25n-649) - p(25n-1276) + p(25n-3751) + \cdots$
$\qquad - 25(n^3-n)\,\sigma_3(n) \equiv 0 \pmod{125}.$

37′

$p(199)$ is the coefficient A_{7^2} in (21.2)

$p(199) = 5^2.63.12195 + 5^4.52.60541 + 5^7.63.66862 + 5^{10}.6.24575$
$\qquad + 5^{11}.6448 = 3646072432125$

$7^2.2526,\ 7^4.48.934,\ 7^5.1418989,\ 7^8.335400$

$\{p(47)x^3 + \cdots\}\{(1-x^7)(1-x^{28})\cdots\}$
$\qquad = 7\sum_1^\infty \{22n^4\sigma_0(n) - 21n^2\sigma_1(n) - \tau(n)\}x^n + 7^3 U.$

(38 continued)

Moduli $5^4, 5^5, \cdots$

(39)

22. Changing again x to x^{24} in (21.1) and arguing as before using (20.5) and (20.7) we can shew that

(22.1) $\qquad p(99) + p(224)x + p(349)x^2 + p(474)x^3 + \cdots$
$\qquad = \sum_1^{5^{24}} a_r \dfrac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{6r-1}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{6r}}$

where the a's are positive integers such that $a_1 = p(99) = 5^3.135^2.839$ and $a_2, a_3, a_4, \cdots$ contain higher powers of 5 than a_1 as factors. It is easy to see from this that

(22.2) $\qquad p(99) + p(224)x + p(349)x^2 + \cdots = 4.5^3\{(1-x)(1-x^2)(1-x^3)\cdots\}^{19} + 5^4 J.$

In this way arguing as before, we can shew that if λ be any positive odd integer, then

(22.3) $\qquad \sum_0^\infty p\left(\frac{19.5^{\lambda+1}}{24} + 5^\lambda n\right)x^n = \sum_1^{5^{\lambda+1}} a_r \dfrac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{6r-1}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{6r}}$

where the a's are positive integers such that $a_1, a_2, a_3, \cdots$ contain higher powers of 5 than a_1 as factors; and if λ be a positive even integer then

(22.4) $\qquad \sum_0^\infty p\left(\frac{23.5^{\lambda+1}}{24} + 5^\lambda n\right)x^n = \sum_1^{5^{\lambda+1}} a_r \dfrac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^{6r}}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^{6r+1}}$

where the a's have the same properties as before. We deduce from (22.3) and (22.4) that if λ is a positive odd integer then

(22.5) $\qquad \sum_0^\infty p\left(\frac{19.5^{\lambda+1}}{24} + 5^\lambda n\right)x^n = c_\lambda . 5^{\lambda}\{(1-x)(1-x^2)(1-x^3)\cdots\}^{19} + 5^{\lambda+1}J$ (40)

and if λ is a positive even integer then

(22.6) $\qquad \sum_0^\infty p\left(\frac{23.5^{\lambda+1}}{24} + 5^\lambda n\right)x^n = c_\lambda . 5^{\lambda}\{(1-x)(1-x^2)(1-x^3)\cdots\}^{23} + 5^{\lambda+1}J.$

where c_λ in both cases is a constant.

We easily deduce from these that if λ is an odd integer greater than 1, Then

$$(22.7)\qquad \begin{cases} p\left(5^{\lambda}n - \frac{5^{\lambda+1}-1}{24}\right) \\ p\left(5^{\lambda}n - \frac{5^{\lambda+1}-1}{24}\right)^{*} \\ p\left(5^{\lambda}n - \frac{49\cdot 5^{\lambda+1}-1}{24}\right) \end{cases} \equiv 0 \pmod{5^{\lambda}}$$

* $\lambda = 1$ may also be 1 in this formula

and if λ is a positive even integer then

$$(22.8)\qquad p\left(5^{\lambda}n - \frac{5^{\lambda}-1}{24}\right) \equiv 0 \pmod{5^{\lambda}}.$$

---

23. We have seen that we can take $c_1 = 1$, $c_2 = -2$, $c_3 = 4$ in (22.5) and (22.6). It appears from § ... that c_λ may probably be some simple function such as $(-2)^{\lambda}$. If we calculate a few more values of c_λ, we can definitely know that it is. Then we can make use of the formulae (22.5) and (22.6) to determine completely the residues of $p\left(5^{\lambda}n - \frac{5^{\lambda+1}-1}{24}\right)$ for odd values of λ and those of $p\left(5^{\lambda}n - \frac{5^{\lambda}-1}{24}\right)$ for even values of λ for moduli $5^{\lambda+1}$. Thus for instance it follows immediately from (22.5) and (22.6) that if λ is an odd integer then (41)

$$\begin{aligned}
&p\left(5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 5^{\lambda}c_\lambda \\
&p\left(2\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 5^{\lambda}c_1 \\
&p\left(3\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^{\lambda}c_1 \\
&p\left(4\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) + 2\cdot 5^{\lambda}c_1 \\
&p\left(5\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) \\
&p\left(6\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^{\lambda}c_\lambda \\
&p\left(7\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^{\lambda}c_\lambda \\
&p\left(8\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 2\cdot 5^{\lambda}c_\lambda \\
&p\left(9\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) \\
&p\left(10\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) \\
&p\left(11\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) + 5^{\lambda}c_1 \\
&p\left(12\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) + 5^{\lambda}c_\lambda \\
&p\left(13\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) - 5^{\lambda}c_1 \\
&p\left(14\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) + 5^{\lambda}c_\lambda \\
&p\left(15\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right) \\
&p\left(16\cdot 5^{\lambda} - \tfrac{5^{\lambda+1}-1}{24}\right)
\end{aligned}$$

and so on are all multiples of $5^{\lambda+1}$; and if λ is an even integer this integer then

$$
\begin{aligned}
&p(5^\lambda - \tfrac{5^\lambda-1}{24}) - 5^\lambda c_\lambda^* \\
&p(2\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) - 2\,5^\lambda c_\lambda^* \\
&p(3\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(4\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(5\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(6\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) - 5^\lambda c_\lambda \\
&p(7\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(8\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(9\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(10\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(11\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) - 5^\lambda c_\lambda \\
&p(12\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) + 5^\lambda c_\lambda \\
&p(13\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(14\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(15\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) \\
&p(16\cdot 5^\lambda - \tfrac{5^\lambda-1}{24}) - 5^\lambda c_\lambda
\end{aligned}
\qquad (42)
$$

and so on are all multiples of $5^{\lambda+1}$

Moduli 7 and 49

24. It is easy to see from (20.1) that

$$\frac{(1-x^{\frac{1}{7}})(1-x^{\frac{2}{7}})(1-x^{\frac{3}{7}})\cdots}{(1-x)(1-x^{11})(1-x^{21})\cdots} = J_1 + x^{\frac{1}{7}}J_2 - x^{\frac{2}{7}}J_1 + x^{\frac{3}{7}}J_3$$

Now cubing both sides we obtain

$$
\begin{aligned}
\frac{1-3x^{\frac{1}{7}}+5x^{\frac{2}{7}}-7x^{\frac{3}{7}}+\cdots}{1-3x^{7}+5x^{14}-7x^{21}+\cdots} &= (J_1^3+3J_2^2J_3x - 6J_1J_2x) + x^{\frac{1}{7}}(3J_1^2J_2 - 3J_2^2J_3x) \\
&\quad + 3x^{\frac{2}{7}}(J_1J_2^2 - J_2^2 + J_3x) + x^{\frac{3}{7}}(J_2^3 - 6J_1J_2 + 3J_1J_3^2x) \\
&\quad + 3x^{\frac{4}{7}}(J_1 - J_2^2 + J_3^2x) + 3x^{\frac{5}{7}}(J_2 + J_1^2J_3 - J_3^2x) \\
&\quad + x^{\frac{6}{7}}(J_1J_2J_3 - 1)
\end{aligned}
\qquad (6J_1J_3x + J_3^2x^2)
$$

But it is easy to see that

$$\frac{1-3x^{\frac{1}{7}}+5x^{\frac{2}{7}}-7x^{\frac{3}{7}}+\cdots}{1-3x^{7}+5x^{14}-7x^{21}+\cdots} = C_1 + x^{\frac{1}{7}}C_2 + x^{\frac{2}{7}}C_3 - 7x^{\frac{6}{7}} \qquad (43)$$

Hence

$$
\begin{cases}
J_1J_2^2 - J_1^2 + J_3x = 0 \\
J_2 - J_2^2 + J_2J_3^2x = 0 \\
J_3 + J_1^2J_3 - J_3^2x = 0 \\
6J_1J_2J_3 - 1 = -7
\end{cases}
$$

All these four equations give the two independent relations:

$$J_1J_2J_3 = -1, \qquad \frac{J_1^2}{J_3} + \frac{J_2}{J_3^2} = x.$$

Now writing ( ) in the form

$$\frac{(1-\omega x^{\frac{1}{7}})(1-\omega^2x^{\frac{1}{7}})(1-\omega^3x^{\frac{1}{7}})\cdots}{(1-x^7)(1-x^{14})(1-x^{21})\cdots} = J_1 + \omega x^{\frac{1}{7}}J_2 - \omega^2x^{\frac{2}{7}}+\omega^3x^{\frac{3}{7}}J_3$$ then $\omega^7 = 1$:

Again writing the seven values of ω in ( ) and multiplying them together and using ( ) we can show that

$$J_1^7 + J_2^7 + J_3^7 + 7x = \left\{\frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}\right\}^8 + 14x\left\{\frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}\right\}^4 + 57x^2$$

$$J_1^2J_2^2 + J_2^2J_3^2 + J_3^2J_1^2 + x^2 = -\left\{\frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}\right\}^4 - 8x$$

$$J_1^3J_2^3 + J_2^3J_3^3 + J_3^3J_1^3 + x^2 = -\left\{\frac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^7)(1-x^{14})(1-x^{21})\cdots}\right\}^4 - 5x$$

Again taking the reciprocals of both sides in ( ) and (44) rationalizing the denominator by using as in (20), we can show that

$$p(5) + p(12)x + p(19)x^2 + p(26)x^3 + \cdots$$
$$= 7\left\{\frac{(1-x^2)(1-x^{14})(1-x^{28})\cdots}{(1-x)(1-x^7)(1-x^{21})\cdots}\right\}^3 + 49x\left\{\frac{(1-x^7)(1-x^{14})(1-x^{21})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\right\}^8.$$

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Other Unpublished Material including portions of Manuscripts

If $x\frac{d}{dx}\log\{(1-x)(1-x^2)(1-x^3)\cdots\}^{12} = \sum_1^\infty \phi(n)x^n$

then $\sum_1^\infty \frac{\phi(n)}{n^s} = \prod \frac{1}{1-\phi(p)p^{-s}+p^{11-2s}}$, (p odd prime).

If $x\frac{d}{dx}\log\{(1-x)(1-x^2)\cdots\}\{(1-x')(1-x'^2)\cdots\}\cdots = \sum_1^\infty \phi(n)x^n$

then $\sum_1^\infty \frac{\phi(n)}{n^s} = \frac{1}{1+3^{-s}+3^{1-2s}}\cdot\frac{1}{1+5^{-s}+5^{1-2s}}\cdot\frac{1}{1+23^{-s}+23^{1-2s}}\cdots$

$\times\frac{1}{1-11^{-s}}\times\frac{1}{1-7^{-2s}}\cdot\frac{1}{1-13^{-2s}}\cdot\frac{1}{1-17^{-2s}}\cdot\frac{1}{1-19^{-2s}}\cdots$

$\times\frac{1}{(1-2^{-s})^2}\cdot\frac{1}{(1-3^{-s})^2}\cdot\frac{1}{(1-5^{-s})^2}$

3, 5, 23 $\cdots \equiv$ ~~1, 3, 4, 5, 9, 0~~ 1, 3, 4, 5, 9 (mod 11),

7, 13, 17 $\cdots \equiv$ 2, 6, 7, 8, 10 (mod 11),

47, 53, 103 $\cdots$ are of the form $11A^2+B^2$.

If $x^{\frac{1}{3}}\{(1-x)(1-x^2)\cdots\}\{(1-x^7)(1-x^{14})\cdots\} = \sum_1^\infty \phi(n)x^{\frac{n}{3}}$,

then $\sum_1^\infty \frac{\phi(n)}{n^s} = \frac{1}{1+2^{-2s}}\cdot\frac{1}{1-3^{-2s}}\cdot\frac{1}{1+11^{-2s}}\cdot\frac{1}{1-17^{-2s}}\cdots$

$\times\frac{1}{1+7^{-s}}\times\frac{1}{(1+3^{-s})^2}\cdot\frac{1}{(1+43^{-s})^2}\cdot\frac{1}{(1-67^{-s})^2}\cdot\frac{1}{(1-79^{-s})^2}$

37, 43, 67 $\cdots \equiv$ 1, 4, 16 (mod 21).

37, 43 are of the form $3A^2+7B^2$ (+ sign).

67, 79 are of the form A^2-6B^2 (− sign),

2, 5, 11, $\cdots \equiv$ ~~2, 5, 0~~ −1 (mod 3) of which

2, 11, $\cdots \equiv$ 2, 8, 11 (mod 21) & 5, 17, $\cdots \equiv$ 5, 17, 20 (mod 21).

A Sheet from the 'LOST NOTEBOOK'

A Sheet from the 'LOST NOTEBOOK'

$$(x^6;x^6)_\infty \{(1-x)(1-x^3)\cdots\}^{-1}\{1-5x^4(\tfrac{1+x}{1-x}+\cdots)\} = \sum_1^\infty \phi(n)x^{\frac{n^2}{2}}$$

$$\text{then } \sum_1^\infty \frac{\phi(n)}{n^s} = \prod_p \frac{1}{1-\phi(p)p^{-s}+p^{7-2s}},$$

p assuming prime values greater than 3,

$$\phi(p) = 0 \qquad p \equiv -1 \pmod 6,$$

$$\phi(p) = (A+iB\sqrt{3})^7 + (A-iB\sqrt{3})^7$$

where A and B are same as before.

77

Add the a 94[15]

$$\int_0^\infty e^{-x^2}\phi(x)\,dx = \psi(y)$$

$$\int_0^\infty e^{-\lambda x}\phi(x)\,dx = \int_0^\infty \lambda^{x-1}$$

$$\int_{-\infty}^{+\infty} \psi(a,\lambda+c\pi)\,\psi(b,\mu+c\pi)$$

A Sheet from the 'LOST NOTEBOOK'

4

Now $\int_{\xi}^{t} t^{k} \alpha^{k-1}(\lambda_0-\lambda)^{n} dt$

$= \left[t^{k} \alpha^{k-1}(\lambda_0-\lambda)^{n} - k t^{k-1} \alpha^{k-2}(\lambda_0-\lambda)^{n+1} + \cdots \right.$

$\left. + (-1)^{k} k!(\lambda_0-\lambda)^{n+k} \right]_{\xi}^{t}$

The terms corresponding to ξ reduce to

$$\xi^{k}\left[\alpha^{k-1}(\lambda_0-\lambda)^{n}\right]_{t=\xi}$$

$$= K\, \xi^{k}(\lambda_0')^{n}$$

Thus we need not bother about these

The terms corresp to ξ are of the

type

(i) $K\left[\lambda_0-\lambda(\xi)\right]^{n}$: this gives something

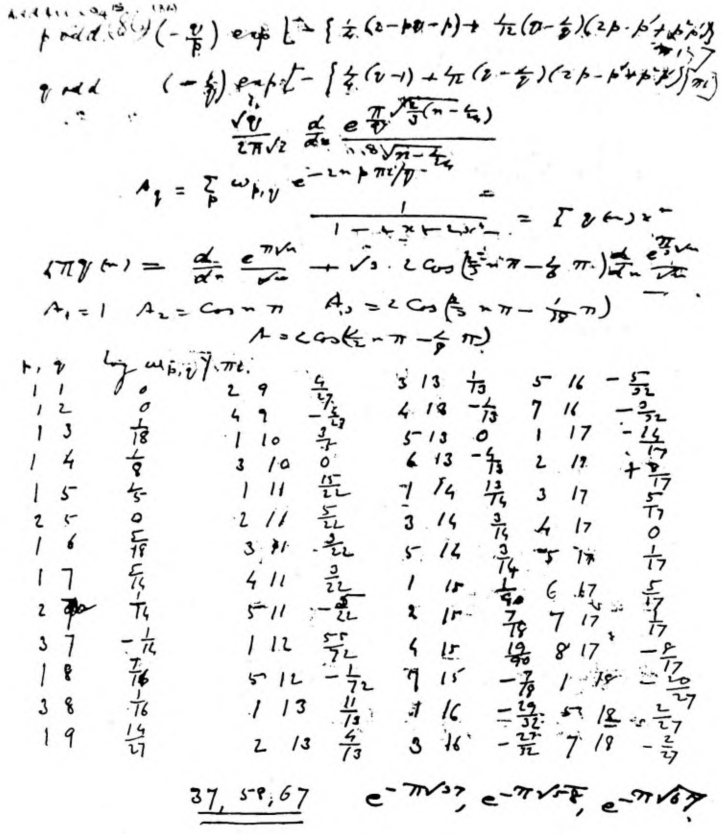
small, as we have ξ multiply λ_0

$\xi \lambda_0^{n-1}$.

(ii) $K \int_{\xi}^{t} t^{k} \alpha^{k-1}(\lambda_0-\lambda)^{n}$ $\quad k>0$.

// This term is of the form

$K_{\xi} \lambda_0^{n-k}$



Add in a $\alpha^{\frac{1}{2}(s-1)}$

Then, ... connected with the ... of ...

If

$$\int_0^\infty x^{s-1} e^{-x^2 - \frac{\alpha^2}{x^2}} dx = \phi_s(\alpha)$$

then

$$\alpha^{\frac{1-s}{2}} \left\{ \sigma_{-s}(1)\,\phi_s(\alpha) + \sigma_{-s}(2)\,\phi_s(2\alpha) + \sigma_{-s}(3)\,\phi_s(3\alpha) + \cdots \right\}$$

$$- \beta^{\frac{1-s}{2}} \left\{ \sigma_{-s}(1)\,\phi_s(\beta) + \sigma_{-s}(2)\,\phi_s(2\beta) + \sigma_{-s}(3)\,\phi_s(3\beta) + \cdots \right\}$$

$$= \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{\frac{1}{2}s} - \alpha^{\frac{1-s}{2}}\right\} + \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{\frac{1+s}{2}} - \alpha^{\frac{1+s}{2}}\right\}$$

provided that $\alpha\beta = \pi^2$, $\sigma_s(n)$ being the sum of the s^{th} powers of the divisors of n.

If $s=1$, the above formula reduces to the known formula

$$\left\{\sigma_{-1}(1)\, e^{-2\alpha} + \sigma_{-1}(2)\, e^{-4\alpha} + \sigma_{-1}(3)\, e^{-6\alpha} + \cdots\right\}$$

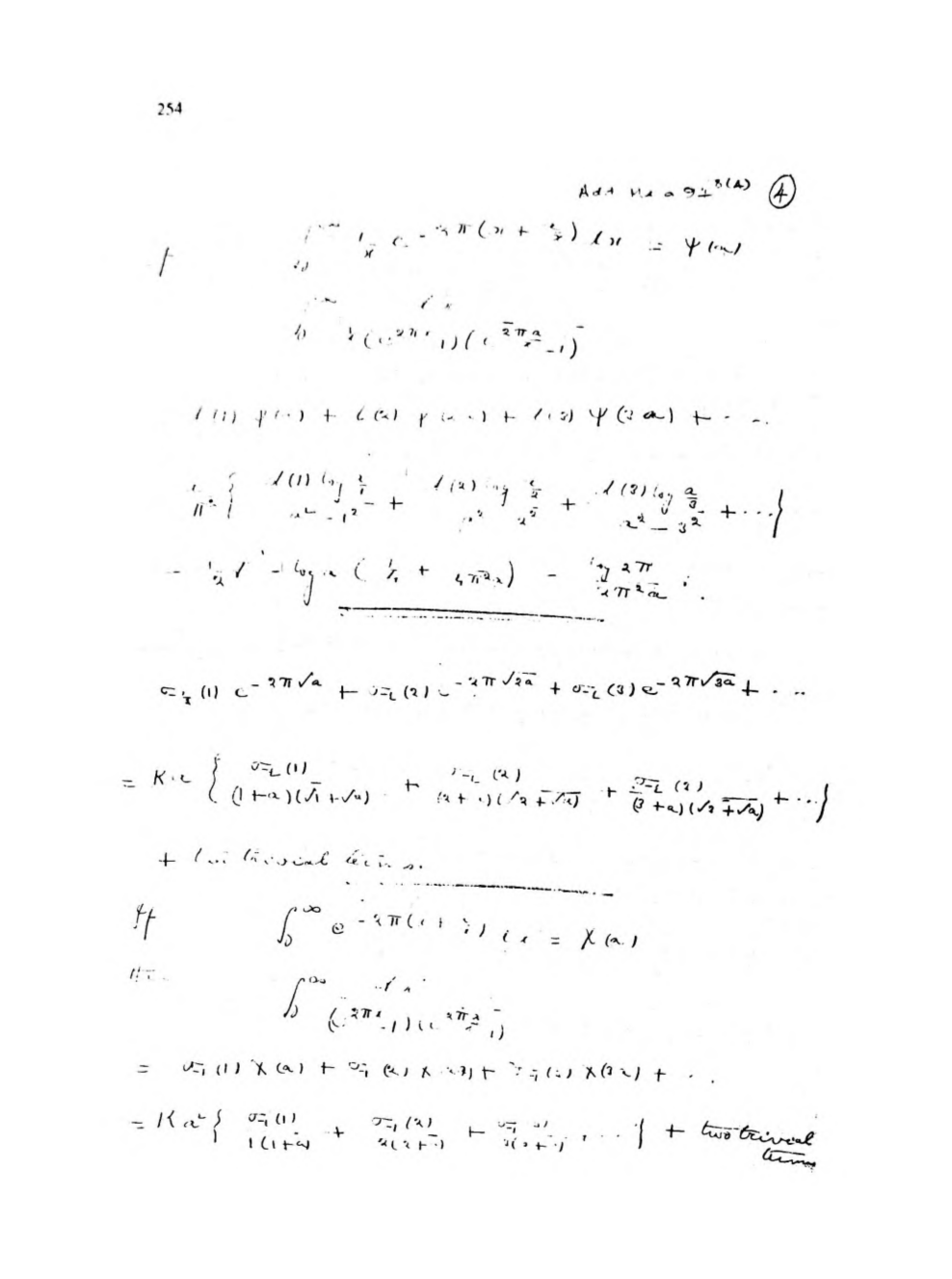
$$- \left\{\sigma_{-1}(1)\, e^{-2\beta} + \sigma_{-1}(2)\, e^{-4\beta} + \sigma_{-1}(3)\, e^{-6\beta} + \cdots\right\}$$

$$= \cdots\cdots$$

with the condition $\alpha\beta = \pi^2$. If $s = 0$ we see that if $\alpha\beta = \pi^2$, then

$$\sqrt{\alpha}\left\{\tfrac{1}{4}\gamma - \tfrac{1}{2}\log 4\beta + d(1)\,\phi_0(\alpha) + d(2)\,\phi_0(2\alpha) + d(3)\,\phi_0(3\alpha) + \cdots\right\}$$

$$= \sqrt{\beta}\left\{\tfrac{1}{4}\gamma - \tfrac{1}{2}\log 4\alpha + d(1)\,\phi_0(\beta) + d(2)\,\phi_0(2\beta) + d(3)\,\phi_0(3\beta) + \right.$$



$$1^s \sigma_n(1) + 2^s \sigma_n(2) + 3^s \sigma_n(3) + \cdots + m^s \sigma_n(m)$$

lies between

$$\zeta(-s)\zeta(n-s) + \frac{m^{1+s}}{1+s}\zeta(1-n) + \frac{m^{1+n+s}}{1+n+s}\zeta(1+n)$$

$$+\tfrac{1}{2}m^s\zeta(-n) + \tfrac{1}{2}m^{n+s}\zeta(n) + \frac{m^{s+\frac{n+1}{2}}}{1-n}$$

and

$$\zeta(-s)\zeta(n-s) + \frac{m^{1+s}}{1+s}\zeta(1-n) + \frac{m^{1+n+s}}{1+n+s}\zeta(1+n)$$

$$+ \tfrac{1}{2}m^s\left\{2\zeta(1-n) - \zeta(-n)\right\} + \tfrac{1}{2}m^{n+s}\left\{2\zeta(1+n) - \zeta(n)\right\} - \frac{m^{s+\frac{n+1}{2}}}{1-n}.$$

$$m^\alpha\left\{1^{s-\alpha}\sigma_n(1) + 2^{s-\alpha}\sigma_n(2) + 3^{s-\alpha}\sigma_n(3) + \cdots + m^{s-\alpha}\sigma_n(m)\right\}$$

$$-\left\{1^s\sigma_n(1) + 2^s\sigma_n(2) + 3^s\sigma_n(3) + \cdots + m^s\sigma_n(m)\right\}$$

$$= m^\alpha\zeta(\alpha-s)\zeta(\alpha-n-s) - \zeta(-s)\zeta(n-s) + \frac{\alpha m^{1+s}}{(1+s)(1-\alpha+s)}\zeta(1-n) + \frac{\alpha m^{1+n+s}\zeta(1+n)}{(1+n+s)(1-\alpha+n+s)}$$

$$-\frac{\alpha}{2}m^{s-1}\sum_1^L\left(\frac{1}{6} - \varepsilon_m + \varepsilon_m^L\right)\left(m^{n+1} + m_{m^{n-1}}^n\right) + O\{\quad\}$$

which lies between

$$m^\alpha\zeta(\alpha-s)\zeta(\alpha-n-s) - \zeta(-s)\zeta(n-s) + \frac{\alpha m^{1+s}\zeta(1-n)}{(1+s)(1-\alpha+s)} + \frac{\alpha m^{1+n+s}\zeta(0+n)}{(1+n+s)(1-\alpha+n+s)}$$

$$-\frac{\alpha}{12}m^{s-1}\zeta(n-1) - \frac{\alpha}{12}m^{n+s-1}\zeta(n-1) - \frac{\alpha m^{s+\frac{n}{2}}}{3(4-n^2)}$$

and

$$m^\alpha\zeta(\alpha-s)\zeta(\alpha-n-s) - \zeta(-s)\zeta(n-s) + \frac{\alpha m^{1+s}\zeta(1-n)}{(1+s)(1-\alpha+s)} + \frac{\alpha m^{1+n+s}\zeta(1+n)}{(1+n+s)(1-\alpha+n+s)}$$

$$-\frac{\alpha}{24}m^{s-1}\left\{3(1-2^{n-1})\zeta(1-n) - \zeta(n-1)\right\}$$

$$-\frac{\alpha}{24}m^{n+s-1}\left\{3\left(1-\frac{1}{2^{n+1}}\right)\zeta(1+n) - \zeta(n-1)\right\} + \frac{\alpha m^{s+\frac{n}{2}}}{6(4-n^2)}$$

If $x \cos\theta = \int_0^\infty \frac{e^{-xt}}{1+t^2}\,dt$ [illegible] (5)

and $x \sin\theta = \int_0^\infty \frac{t\,e^{-xt}}{1+t^2}\,dt$

whence $e^{x} = \int_0^\infty \frac{e^{-xt}}{t}\log(1+t^2)\,dt$

Then $\int_0^x \frac{\sin u}{u}\,du = \frac{\pi}{2} - x\cos(x-\theta)$

and $\int_0^x \frac{1-\cos u}{u}\,du = \gamma + \log x - x\sin(x-\theta)$.

1st max value of $\int_0^x \frac{\sin u}{u}\,du = \frac{\pi}{2} + \int_0^\infty \frac{e^{-\pi t}}{1+t^2}\,dt$

1st min " " do " " $= \frac{\pi}{2} - \int_0^\infty \frac{e^{-2\pi t}}{1+t^2}\,dt$

2nd max " " " " $= \frac{\pi}{2} + \int_0^\infty \frac{e^{-3\pi t}}{1+t^2}\,dt$

and so on.

Appendix - Results by Ramanujan. Added May 1918 (8/17)

All variables considered in the following results are positive

1. $a_1 + a_2 + \ldots a_n$ can be expressed in terms of n with an error of $O(\max a_n - \min a_n)$. Perhaps we may go even as far as ascertaining the maximum and the minimum of O and it is not possible to go beyond that. e.g.

(i) $d(1) + d(2) + d(3) + \ldots d(n) = n(2\gamma - 1 + \log n) + O(\max d(n))$

$= n(2\gamma - 1 + \log n) + 2^{O\left(\frac{\log n}{\log\log n}\right)}$.

(ii) $\log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \ldots$ to n terms can be expressed in terms of n with an error of $o(1)$ or even $O(n^{-\frac{1}{2}+\varepsilon})$; for

$O(\max \log p_n - \min \log p_n) = O(n^{-\frac{1}{2}+\varepsilon})$.

2. The maximum order of $\sum a_n$ = max. R (average order of a_n^R) for the variable n. e.g.

(i) max. $d(n)$ can be shown to be $2^{\frac{\log n}{\log\log n}(1+\varepsilon)}$

(ii) The maximum order of $\pi(p)$

$= Li(p) + \max. R\{\text{average order of } (p - p')^R\}$

where p' is the prime just less than the prime p.

3. If b_n is steadily increasing and if $\max a_n > b_n$, tend if $(a_1 e^{-x} + a_2 e^{-2x} + a_3 e^{-3x} + \ldots\ldots)$

$-(b_1 e^{-x} + b_2 e^{-2x} + b_3 e^{-3x} + \ldots\ldots) = O(1)$ as $x \to 0$

then $a_1 + a_2 + a_3 + \ldots a_n$

$= b_1 + b_2 + b_3 + \ldots\ldots b_n + O(\max a_n - \min a_n)$

4. If in (3) a_n also is steadily increasing, then

$a_n \sim b_n$.

Proc. Lon. Math. Soc. (2) 24 (1926)

5. Analogous results in case of

$$\frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots$$

(i) e.g. The no of nos of the form $2^m 3^n$ such as 1, 2, 3, 4, 6, 8, 9, 12, ... less than $x = \frac{1}{2} \cdot \frac{\log(2x)}{\log 2} \cdot \frac{\log(3x)}{\log 3} + O(1)$.

(ii) No of the form a^2+b^2 less than x

$$= C \int \frac{dx}{\sqrt{\log x}} + O(x^{\frac{1}{2}+\varepsilon})$$

where $C = \sqrt{2}(1-\frac{1}{3^2})(1-\frac{1}{7^2})(1-\frac{1}{11^2})(1-\frac{1}{19^2})(1-\frac{1}{23^2})(1-\frac{1}{31^2})(1-\frac{1}{43^2})\cdots$

$\sqrt{2}(1-\frac{1}{3^2})(1-\frac{1}{7^2})(1-\frac{1}{11^2})(1-\frac{1}{19^2}) = (1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{19})$.

(iii) No of the form a^2b^3 such as 1, 4, 8, 9, 16, 25, 27, 32, 36, ... less than x

$$= \sqrt{4.7\,9\,3\,0\,3\,4}\, x^{\frac{1}{2}} \cdot \sqrt[3]{3.10227}\, x^{\frac{1}{3}} + O(x^{\frac{1}{6}+\varepsilon}).$$

6. If a_n and b_n are steadily increasing and if

$$\frac{1}{a_1^s} + \frac{1}{a_2^s} + \frac{1}{a_3^s} + \cdots$$

and $\frac{1}{b_1^s} + \frac{1}{b_2^s} + \frac{1}{b_3^s} + \cdots$

are both convergent when $s > k$ and both are divergent when $s = k$ and if the difference between the two series be $O(1)$ when $s = k$, then

$$a_n \sim b_n.$$

7. If a_n and b_n are steadily increasing and if

$$e^{-a_1 x} + e^{-a_2 x} + e^{-a_3 x} + \cdots$$

$$\sim e^{-b_1 x} + e^{-b_2 x} + e^{-b_3 x} + \cdots \quad \text{as } x \to 0$$

Mersenne's Numbers

(1)

When 2^p-1 is a prime, p may be termed as a Mersenne's Prime

(1) All Mersennic primes are either of the form a^2+ab+b^2 or of the form a^2+b^2. Then since a number of the form $12n-1$ cannot be expressed in any one of the above two forms, we infer that

(2) A Mersennic prime is never of the form $12n-1$. Thus for example 2^4-1, $2^{23}-1$, $2^{47}-1$, $2^{59}-1$, $2^{71}-1$, $2^{83}-1$, $2^{107}-1$, $2^{131}-1$, $2^{167}-1$, $2^{179}-1$, $2^{191}-1$, $2^{227}-1$, $2^{239}-1$, $2^{251}-1$, &c should be composite numbers. Hence we may divide all Mersennic primes into two classes, one comprising primes that can be expressed as a^2+ab+b^2 and the other containing primes that cannot be expressed as a^2+ab+b^2.

(3) Hence the Mersennic primes of the 1st class except 1 and 3 are of the form $6n+1$, while those of the 2nd except 2 are of the form $12n+5$. Thus we have,

$$\begin{cases} \text{Nos of the 1st class :- } 1, 3, 7, 13, 19, 31, 61, 127, \text{ \&c} \cdots \\ \text{Nos of the 2nd class :- } 2, 5, 17, 89, 257, \text{ \&c} \cdots \end{cases}$$

(4) Theorem

If P be any prime, and p any odd prime and if either of $\frac{P^p-1}{P-1}$ or $\frac{P^p-1}{P(P-1)}$ happens to be a prime, then that prime will be a Mersennic prime of the 1st class. As a particular case we have when $P=3$,

(5) If P be any prime and if either of P^2+P+1 or $\frac{P^2+P+1}{3}$ happens to be a prime, then that prime will be a Mersennic prime of the 1st class. As another particular case when $P=2$ we have

(6) If p be a Mersennic prime, then 2^p-1 will be a Mersennic prime of the 1st class. As examples of (5) and (6) we have

(2)

$1^2+1+1=3$; $2^2+2+1=7$; $3^2+3+1=13$; $5^2+5+1=31$;

$\frac{7^2+7+1}{3}=19$; $(11^2+11+1=133$ composite$)$; $\frac{13^2+13+1}{3}=61$;

$17^2+17+1=307$; $\frac{19^2+19+1}{3}=127$; and so on. Again

$2^2-1=3$; hence $2^3-1=7$ a prime ; hence $2^7-1=127$ a prime ;

hence $2^{127}-1$ is a prime. $2^5-1=31$; hence $2^{31}-1$ is a prime.

(7) From (3) we can infer that, Mersenne's primes of the 2nd class is always about $\frac{1}{3}$ of the number of the Mersenne's primes of the 1st class. There may be a general theorem like (4) for the Mersenne's primes of the 2nd class of which the particular case analogous to (6) will be,

(8) If 2^b+1 be a prime, then 2^b+1 will be a Mersenne's prime of the 2nd class. Thus for example we have

$2+1=3$; hence 2^3-1 is a prime ; $2^2+1=5$ hence

2^5-1 is a prime ; $2^4+1=17$ hence $2^{17}-1$ is a prime

$2^8+1=257$; hence $2^{257}-1$ is a prime and so on.

(9) Mersenne's primes of the 2nd class are always of the form $(2^a)^2+(4b+1)^2$ where a assumes all integral values, 0, 1, 2, 3 &c without an exception, b is a positive integer including 0, $4b+1$ is never greater than 2^a and for every value of a, there is at least one value of b. Thus we have, when $a=0$, $b=0$ hence 2^2-1 is a prime; when $a=1$, b must be 0 hence 2^5-1 is a prime ; when $a=2$, b must be 0 hence $2^{17}-1$ is a prime ; when $a=3$, b may be 0 or 1, but when $b=0$ $(2^3)^2+1$ becomes composite, hence b must be 1 so that since when $(2^3)^2+5^2$ is a prime, hence $2^{89}-1$ is a prime.

(10) Another theorem analogous to (8) is, if 2^b+1 is a prime then $2^{2^b}+1$ is also a prime. Hence we have

This completes the MSS. Mus S had, except for
(1) sheet 1 mentioned before
(2) two sheets MSS studies
$\Pi_{\nu}^{(1)}$, $\Phi_{\lambda}^{(1)}$ — connected with
Paper No 35.
These 1 thought Mus S had — letter
shall see

Love.

Add. Note on p. 18 (13)

Paper a little difficult to understand after 1st page. I don't see where eqn (12) comes from, & $m = 5$, $k = 8$ (quoted on p. 2) does not come from the values of k given $[\frac{3}{5}]$, as x is $-\frac{1}{6}$.

Add. Note on p. 18 (12) (12)

Add. problem. Trans- parent throw. Matter Hou's work in it

Let us consider the maximum of

(1) $$\varepsilon_m (1-\varepsilon_m)(1-2\varepsilon_m)$$

when ε_m is a positive proper fraction and m and $m\varepsilon_m$ are positive integers. Let v_m be the maximum of (1). If we do not assume that $m\varepsilon_m$ is ~~rational integer~~, we get that

(2) $$\varepsilon_m = \frac{3-\sqrt{3}}{6} \quad \text{or} \quad v_m = \frac{1}{6\sqrt{3}}$$

Now let $m\left(\frac{3-\sqrt{3}}{6}\right) - \varepsilon$ be an integer where ε is a positive proper fraction. Then we see from (2) that ε_m is either

(3) $$\frac{m\cdot\left(\frac{3-\sqrt{3}}{6}\right)-\varepsilon}{m} \quad \text{or} \quad \frac{m\cdot\left(\frac{3-\sqrt{3}}{6}\right)+1-\varepsilon}{m}$$

(4) If $\varepsilon_m = \frac{m\left(\frac{3-\sqrt{3}}{6}\right)-\varepsilon}{m}$ then $v_m = \frac{1}{6\sqrt{3}} - \frac{\varepsilon^2}{m^2}\sqrt{3} - 2\frac{\varepsilon^3}{m^3}$

(5) and if $\varepsilon_m = \frac{m\cdot\left(\frac{3-\sqrt{3}}{6}\right)+1-\varepsilon}{m}$, then $v_m = \frac{1}{6\sqrt{3}} - \frac{(1-\varepsilon)^2}{m^2}\sqrt{3} + 2\cdot\frac{(1-\varepsilon)^3}{m^3}$

(6) Hence if $\varepsilon < \frac{1}{2} - \frac{m-\sqrt{(m^2-1)}}{2\sqrt{3}}$, (4) is greater

(7) and if $\varepsilon > \frac{1}{2} - \frac{m-\sqrt{(m^2-1)}}{2\sqrt{3}}$, (5) is greater.

Hence from (3), (6) and (7) we have

(8) $$\varepsilon_m = \frac{\left[\frac{m+1}{2} - \sqrt{\left(\frac{m^2-1}{12}\right)}\right]}{m}$$

Hence we have

$$v_1 = 0,\ v_2 = 0,\ v_3 = \frac{2}{3^3},\ v_4 = \frac{6}{4^3},\ v_5 = \frac{12}{5^3},\ v_6 = \frac{20}{6^3},$$

$$v_7 = \frac{30}{7^3},\ v_8 = \frac{48}{8^3},\ v_9 = \frac{70}{9^3},\ v_{10} = \frac{96}{10^3},\ \ldots\ldots$$

Add to a + '3 (13)

From (4) and (5) we see that the order of ν_m is minimum when $\varepsilon = \frac{1}{2} - \frac{m - \sqrt{(m^2-1)}}{2\sqrt{3}}$ that is the minimum order

(9) of $\nu_m = \frac{(m^2-4)}{6m^3}\sqrt{\left(\frac{m^2-1}{3}\right)}$ and the corresponding value of $\varepsilon_m = \frac{\frac{m+1}{2} - \sqrt{\left(\frac{m^2-1}{12}\right)}}{m}$.

The maximum order of ν_m can be got by making ε as small as possible in (4), that is by making $m\varepsilon_m$ as nearly equal to $m\left(\frac{3-\sqrt{3}}{6}\right)$ as possible from below.

(10) Hence $\varepsilon_m = \frac{\frac{m}{2} - \sqrt{\left(\frac{m^2+2}{12}\right)}}{m}$

and $\nu_m = \frac{(m^2-1)}{6m^3}\sqrt{\left(\frac{m^2+2}{3}\right)}$.

1. From the table we find that

$$\nu_5 = \nu_{10} = \nu_{15} = \nu_{20} = \nu_{25} = \nu_{30} = \nu_{35} = \nu_{40}.$$

which suggests the question

how to determine the maximum values of k in order that

(11) $$\nu_m = \nu_{2m} = \nu_{3m} = \cdots\cdots = \nu_{km}$$

From (9) and (10) we see that

$$k \neq \left[\frac{x}{m}\right]$$

where $\frac{\frac{x+1}{2} - \sqrt{\left(\frac{x^2-1}{12}\right)}}{x} = \frac{\frac{m}{2} - \sqrt{\left(\frac{m^2+3}{12}\right)}}{m}$. That is

(12) $$k \neq \left[\sqrt{(3m^2+6)} - 1\right]$$

It can also be shown that, if $3m^2+6$ is a perfect square

(13). then $k = \sqrt{(3m^2+6)} - 1$

The actual value of k can be got from (8) and (9). Thus

$$k = \left[\frac{x}{m}\right]$$

(14)

where $$\frac{\frac{x+1}{2} - \sqrt{\left(\frac{x^2-1}{12}\right)}}{x} = \frac{\left[\frac{m+1}{2} - \sqrt{\left(\frac{m^2-1}{12}\right)}\right]}{m}.$$

Thus for example we have

$$v_5 = v_{10} = v_{19} = \ldots = v_{40}$$

$$v_9 = v_{18} = v_{17} = \ldots = v_{45}$$

$$v_{14} = v_{28} = v_{42} = \ldots = v_{168}$$

$$v_{19} = v_{38} = v_{57} = \ldots = v_{608}$$

$$v_{71} = v_{142} = v_{213} = \ldots = v_{8662}$$

$$v_{265} = v_{530} = v_{795} = \ldots = v_{121670}$$

$$v_{989} = v_{1978} = v_{2967} = \ldots = v_{1693168}$$

and so on. The irregularity of v_m can also be seen from the following example

$$v_{989} = \frac{1}{6\sqrt{3}} - \frac{1}{\sqrt{3}} \cdot \frac{(1978 + 571\sqrt{3})}{(1713 + 989\sqrt{3})^2} \cdot \frac{1}{989^3}$$

$$= \frac{1}{6\sqrt{3}} - \frac{3}{2.10^{13}} \text{ nearly.}$$

while

$$v_{1351} = \frac{1}{6\sqrt{3}} - \frac{3}{2.10^{13}} \cdot \frac{(2342 + 1351\sqrt{3})}{}$$

$$= \frac{1}{6\sqrt{3}} - \frac{1}{4.10^6} \text{ nearly.}$$

$$v_\infty = \frac{1}{6\sqrt{3}} = .0962250$$

$v_1 = 0$

$v_2 = 0$

$v_3 = .0740741$

$v_4 = .0937500$

$v_5 = .0960000$ ✓

$v_6 = .0925926$

$v_7 = .0874636$ ✓

$v_8 = .0937500$

$v_9 = .0960219$

$v_{10} = .0960000$

$v_{11} = .0946657$

$v_{12} = .0937500$

$v_{13} = .0955849$

$v_{14} = .0962099$

$v_{15} = .0960000$

$v_{16} = .0952148$

$v_{17} = .0952575$

$v_{18} = .0960219$

$v_{19} = .0962239$

$v_{20} = .0960000$

$v_{21} = .0954541$

$v_{22} = .0957926$

$v_{23} = .0961618$

$v_{24} = .0962095$

$v_{25} = .0960000$

$v_{26} = .0955849$

$v_{27} = .0960219$

$v_{28} = .0962099$

$v_{29} = .0961905$

$v_{30} = .0960000$

$v_{31} = .0958679$

$v_{32} = .0961304$

$v_{33} = .0962239$

$v_{34} = .0961734$

$v_{35} = .0960000$

$v_{36} = .0960219$

$v_{37} = .0961838$

$v_{38} = .0962239$

$v_{39} = .0961581$

$v_{40} = .0960000$

$v_{41} = .0961100$

$v_{42} = .0962099$

$v_{43} = .0962179$

$v_{44} = .0961448$

$v_{45} = .0960219$

$v_{46} = .0961618$

$v_{47} = .0962215$

$v_{48} = .0962095$

$v_{49} = .0961334$

$v_{50} = .0960960$

Add to a §4.[13] (16)

Apparently very incomplete ~~(Does this follow for later ...)~~

(10) ? & what diff does |a| make?

(13) & (16) ?

$v_2 - v_1$: even v_1 odd : v_2 odd
$v_4 - v_2$ odd v_2 even : v_4 odd
$v_5 - v_3$ odd v_3 odd : v_5 even

Add to add[n]

See §4.7 §4(ii) in Virtual Thin [illegible]

Let $[x]$ denote the greatest integer in x. Notation.

I *a is any integer*

It is possible to find an integer N as large as we please such that

$$(1) \qquad Ne^{\frac{2\pi}{a}} - \left[Ne^{\frac{2\pi}{a}}\right] < \frac{(1+\varepsilon)\log\log N}{|a|\, N\log N}$$

however small ε may be.

Given ε, there is a δ such that

$$(2) \qquad Ne^{\frac{2\pi}{a}} - \left[Ne^{\frac{2\pi}{a}}\right] > \frac{(1-\varepsilon)\log\log N}{|a|\, N\log N}$$

for all values of N greater than δ.

II *a is any even integer*

It is possible to find an integer N as large as we please such that

$$(3) \qquad 1 + \left[Ne^{\frac{2\pi}{a}}\right] - Ne^{\frac{2\pi}{a}} < \frac{(1+\varepsilon)\log\log N}{|a|\, N\log N}.$$

Given ε, there is a δ such that

$$(4) \qquad 1 + \left[Ne^{\frac{2\pi}{a}}\right] - Ne^{\frac{2\pi}{a}} > \frac{(1-\varepsilon)\log\log N}{|a|\, N\log N}$$

for all values of N greater than δ.

III *a is any odd integer*

It is possible to find N as large as we please such that

$$(5) \qquad 1 + \left[Ne^{\frac{2\pi}{a}}\right] - Ne^{\frac{2\pi}{a}} < \frac{(1+\varepsilon)\log\log N}{4\,|a|\, N\log N}.$$

Given ε, there is a δ such that

$$1 + \left[Ne^{\frac{2\pi}{a}}\right] - Ne^{\frac{2\pi}{a}} > \frac{(1-\varepsilon)\log\log N}{4\,|a|\, N\log N}$$

for all values of N greater than δ.

Proof Taking the equation

Add [illegible] a 9th [illegible] (17)

$$\tanh x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\,\frac{x^2}{7+}\cdots\cdots$$

we see that

$$(6)\qquad \frac{1}{2}\left(e^{\frac{2}{a}}+1\right) = \frac{1}{1-}\,\frac{1}{a+}\,\frac{1}{3a+}\,\frac{1}{5a+}\,\frac{1}{7a+}\cdots\cdots$$

Now let

$$(7)\qquad \begin{cases} u_1 = 1, & v_1 = 1; \\ u_2 = |a|, & v_2 = |a-1|; \end{cases}$$

and

$$(8)\qquad \begin{cases} u_{n+1} - u_{n-1} = (2n-1)|a|u_n; \\ v_{n+1} - v_{n-1} = (2n-1)|a|v_n. \end{cases}$$

Then we have

$$(9)\qquad \frac{1}{2}\left(e^{\frac{2}{a}}+1\right) = \frac{u_n}{v_n} + (-1)^{n-1}\left(\frac{1}{v_n v_{n+1}} - \frac{1}{v_{n+1}v_{n+2}} + \cdots\cdots\right).$$

It is easy to deduce from (8) that

$$(10)\qquad \begin{cases} v_{n+1} \sim 2|a|\,n v_n, \\ \log v_n \sim n\log n. \end{cases}$$

It follows from (9) and (10) that

$$(11)\qquad e^{\frac{2}{a}}+1 - \frac{2u_n}{v_n} \sim \frac{(-1)^{n-1}}{|a|\,n v_n^2}.$$

If a is even, v_n is always odd. In this case choosing $N = v_n$ so that $n \sim \log N/\log\log N$ we see that

$$(12)\qquad N\left(e^{\frac{2}{a}}+1\right) - 2u_n \sim \frac{(-1)^{n-1}\log\log N}{|a|\,N\log N}.$$

If a is odd and n is odd, v_n is again odd. Let $N = v_n$. Then

$$(13)\qquad N\left(e^{\frac{2}{a}}+1\right) - 2u_n \sim \frac{\log\log N}{|a|\,N\log N}.$$

If a is odd and n is even, then v_n is even. Now we can choose $N = \frac{1}{2}v_n$ so that

$$(14)\qquad N\left(e^{\frac{2}{a}}+1\right) - u_n \sim -\frac{\log\log N}{4|a|\,N\log N}.$$

Proof of final formula completed if it is true that-

$$\frac{\Pi(b,x)}{\Pi(a,x)} \Big/ \frac{\Pi(-c,x)}{\Pi(-a,x)} \cdot \frac{\Pi(-b,x)}{\Pi(-a,x)}.$$

assuming $\dfrac{\Pi(a,x)}{\Pi(a,x)} = 1 + \dfrac{a+b}{1-x} + \dfrac{(a+b)(a+bx)}{(1-x)(1-x^2)} + \cdots$

can be proved that-

$$(c-1)(c-x)(c-x^2)\cdots\cdots(c-x^{n-1})$$

$$c^n - c^{n-1}\frac{1-x^n}{1-x} + c^{n-2}x\frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)} - c^{n-3}x^3\frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)(1-x^3)}$$

in descending powers of c

$$\therefore c^n = 1 + (c-1)\frac{1-x^n}{1-x} + (c-1)(c-x)\frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)} +$$

$$(c-1)(c-x)(c-x^2)\cdot\frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)(1-x^3)} + \cdots$$

for the coeff^{ts} of c^n is 1 and all other coeffts can be shewn to be 0 by taking the coeff^{ts} of c^n and applying the result (1).

(3). Now the coeff^{t} of a^n in

$$\frac{\Pi(a,x)}{\Pi(al,x)} + a\cdot\frac{(1-l)(1-c)}{1-x}\cdot\frac{\Pi(ax,x)}{\Pi(al,x)} +$$

$$a^2\cdot\frac{(1-l)(x-l)(1-c)(x-c)}{(1-x)(1-x^2)}\cdot\frac{\Pi(ax^2,x)}{\Pi(al,x)} + \cdots$$

$$= \frac{(l-1)(l-x)\cdots(l-x^{n-1})}{(1-x)(1-x^2)\cdots(1-x^n)} + \frac{(1-l)(1-c)}{1-x}\cdot\frac{(l-x)(l-x^2)\cdots(l-x^{n-1})}{(1-x)(1-x^2)\cdots(1-x^{n-1})}$$

$$+ \frac{(1-l)(x-l)(1-c)(x-c)}{(1-x)(1-x^2)}\cdot\frac{(l-x)(l-x^2)\cdots(l-x^{n-2})}{(1-x)(1-x^2)\cdots}\cdots + \cdots$$

$$= \frac{(l-1)(l-x)\cdots(l-x^{n-1})}{(1-x)(1-x^2)\cdots(1-x^n)}\left\{1 + (c-1)\cdot\frac{1-x^n}{1-x} + (c-1)(c-x)\frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)} + \cdots\right\}$$

$C^n \cdot \frac{(b-1)(b-x)\cdots(b-x^{n-1})}{(1-x)(1-x^2)\cdots(1-x^n)}$ by the result (2).

(4) But the coeff^t of a^n in $\frac{\Pi(ac,x)}{\Pi(abc,x)}$ can be shewn to be equal to $C^n \cdot \frac{(b-1)(b-x)\cdots(b-x^{n-1})}{(1-x)(1-x^2)\cdots(1-x^n)}$ and hence

$$\frac{\Pi(ac,x)}{\Pi(abc,x)} = \frac{\Pi(a,x)}{\Pi(ab,x)} + a\cdot\frac{(1-b)(1-c)}{1-x}\cdot\frac{\Pi(ax,x)}{\Pi(ab,x)}$$

$$+ a^2\cdot\frac{(1-b)(x-b)(1-c)(x-c)}{(1-x)(1-x^2)}\cdot\frac{\Pi(ax^2,x)}{\Pi(ab,x)} + \cdots$$

(5) That is $\frac{\Pi(ab,x)\,\Pi(ac,x)}{\Pi(a,x)\,\Pi(abc,x)}$

$$1 + a\cdot\frac{(1-b)(1-c)}{(1-x)(1-a)} + a^2\frac{(1-b)(x-b)(1-c)(x-c)}{(1-x)(1-x^2)(1-a)(1-ax)} + \cdots$$

Incomplete?

Intended as conclusion to 'Certain Trigonometrical Sums' 21 p. 179.

~~Does not include first~~ ~~$\sum_1^\infty \sigma_s(n)x^n$~~ ~~or~~ ~~$\sum_1^\infty \sigma_s(n)e^{2\pi i n x}$~~ ~~as it sets out to do~~

Last 2 pages occupied on 'A new subject'.

Add MS on D.

(29)

18. I shall conclude this paper by finding an expression for

$$\sum_1^\infty \sigma_s(n)x^n, \quad \sum_1^\infty R_s(n)x^n$$

which shows the asymptotic nature for large values of x. If $R(x) > 0$, $R(x)$ being the real part of x, it is well known that

$$\pi + 2\pi\left(e^{-2\pi x} + e^{-4\pi x} + e^{-6\pi x} + \cdots\right) \tag{18·1}$$

$$= \frac{1}{x} + \frac{1}{x+i} + \frac{1}{x-i} + \frac{1}{x+2i} + \frac{1}{x-2i} + \cdots.$$

Differentiating the two sides in (18·1) $s-1$ times we obtain

$$1^{s-1}e^{-2\pi x} + 2^{s-1}e^{-4\pi x} + 3^{s-1}e^{-6\pi x} + \cdots \tag{18·11}$$

$$= \frac{\Gamma(s)}{(2\pi)^s}\left\{\frac{1}{x^s} + \frac{1}{(x+i)^s} + \frac{1}{(x-i)^s} + \frac{1}{(x+2i)^s} + \cdots\right\}$$

if s is ~~a positive~~ an integer greater than 1. This result is quite elementary. But it can be shown by the theory of residues that (18·11) is true for all values of s greater than 1. It follows from this that

$$\sum_1^\infty \sigma_{s-1}(n)e^{2\pi n x} = \sum_1^\infty\left(1^{s-1}e^{-2\pi n x} + 2^{s-1}e^{-4\pi n x} + \cdots\right)$$

$$= \frac{\Gamma(s)}{(2\pi)^s}\sum_1^\infty\left\{\frac{1}{(nx)^s} + \frac{1}{(nx+i)^s} + \frac{1}{(nx-i)^s} + \cdots\right\}$$

The double series in the last equation is absolutely convergent if $s > 2$. From this we easily deduce that

$$\sigma_{s-1}(1)e^{-2\pi x} + \sigma_{s-1}(2)e^{-4\pi x} + \cdots \tag{18·2}$$

$$= \frac{\Gamma(s)}{(2\pi)^s}\zeta(s)\left\{\frac{1}{x^s} + \sum'\frac{1}{(\mu x+\nu i)^s} + \sum'\frac{1}{(\mu x-\nu i)^s}\right\}$$

$$= \frac{\Gamma(s)}{(2\pi)^s} f(s) \left\{ \frac{1}{x^s} + 2 \sum \frac{\cos\left(s \tan^{-1} \frac{v}{\mu x}\right)}{(\mu^2 x^2 + v^2)^{\frac{1}{2}s}} \right\},$$

where $s > 2$ and μ and ν assume all positive integral values which are prime to each other.

Let us consider some particular cases of (18·2). Suppose $x = 1$. Then

$$2 \sum \frac{\cos\left(s \tan^{-1} \frac{v}{\mu}\right)}{(\mu^2 + v^2)^{\frac{1}{2}s}} = \sum \frac{\cos\left(s \tan^{-1} \frac{v}{\mu}\right) + \cos\left(s \tan^{-1} \frac{\mu}{v}\right)}{(\mu^2 + v^2)^{\frac{1}{2}s}}$$

$$= 2 \cos \frac{\pi s}{4} \sum \frac{\cos\left(s \tan^{-1} \frac{\mu - v}{\mu + v}\right)}{(\mu^2 + v^2)^{\frac{1}{2}s}}.$$

It follows that if $s > 2$, then

$$(18·3) \quad \sigma_{s-1}(1) e^{-2\pi} + \sigma_{s-1}(2) e^{-4\pi} + \sigma_{s-1}(3) e^{-6\pi} + \cdots$$

$$= \frac{\Gamma(s)}{(2\pi)^s} f(s) \left\{ 1 + 2 \cos \frac{\pi s}{4} \sum \frac{\cos\left(s \tan^{-1} \frac{\mu - v}{\mu + v}\right)}{(\mu^2 + v^2)^{\frac{1}{2}s}} \right\}$$

$$= \frac{\Gamma(s)}{(2\pi)^s} f(s) \left\{ 1 + 2 \cos \frac{\pi s}{4} \left(\frac{1}{2^{\frac{1}{2}s}} + \frac{2 \cos\left(s \tan^{-1} \frac{1}{2}\right)}{5^{\frac{1}{2}s}} \right.\right.$$

$$\left.\left. + \frac{2 \cos\left(s \tan^{-1} \frac{1}{4}\right)}{10^{\frac{1}{2}s}} + \frac{2 \cos\left(s \tan^{-1} \frac{1}{3}\right)}{13^{\frac{1}{2}s}} + \cdots \right) \right\},$$

where μ and ν are positive integers that are prime to each other and $2, 5, 10, 13, \cdots$ are sum of two squares that are prime to each other.

Similarly putting $x = \frac{1}{2}(\sqrt{3} + i)$ in (18·2) we can show that if $s > 2$ then

$$(18·4) \quad \sigma_{s-1}(1) e^{-\pi\sqrt{3}} - \sigma_{s-1}(2) e^{-2\pi\sqrt{3}} + \sigma_{s-1}(3) e^{-3\pi\sqrt{3}} - \cdots$$

$$= -\frac{2\Gamma(s)}{(2\pi)^s} f(s) \left\{ \cos \frac{\pi s}{6} + 2 \cos \frac{\pi(s+1)}{6} \cos \frac{\pi(s-1)}{6} \right.$$

$$\left. \sum \frac{\cos\left(s \tan^{-1} \frac{N}{\lambda\sqrt{3}}\right)}{(\mu^2 + \mu\nu + \nu^2)^{\frac{1}{2}s}} \right\}$$

Add to a [illegible]

$$\sigma_s(n) = n^s \sin \pi n \left\{ \frac{1}{1^{s+1} \sin \pi n} + \frac{1}{2^{s+1} \tan \frac{1}{2}\pi n} \right.$$

$$+ \frac{1}{3^{s+1} \sin \frac{1}{3}\pi n} + \frac{1}{4^{s+1} \tan \frac{1}{4}\pi n}$$

$$+ \frac{1}{5^{s+1} \sin \frac{1}{5}\pi n} + \frac{1}{6^{s+1} \tan \frac{1}{6}\pi n}$$

$$\left. + \cdots \right\}$$

$$\left(1 + 2x + 2x^4 + 2x^9 + \cdots\right)^s = \sum R_s(n)\, x^n.$$

If s is an integer greater than 1, then the asymptotic expression of $R_{2s}(n)$ (by the partition method) is

$$\frac{\pi^s n^{s-1}}{(s-1)!} \sin \pi n \left\{ \frac{1}{1^s \sin \pi n} + \frac{1}{2^s \sin\left(\frac{1}{2}\pi n + \frac{1}{2}\pi s\right)} \right.$$

$$+ \frac{1}{3^s \sin\left(\frac{1}{3}\pi n + \pi s\right)} + \frac{1}{4^s \sin\left(\frac{1}{4}\pi n + \frac{3}{2}\pi s\right)} +$$

$$\left. \frac{1}{5^s \sin\left(\frac{1}{5}\pi n + 2\pi s\right)} + \cdots \right\} \div$$

$$\left(1^{-s} + 3^{-s} \cos \pi s + 5^{-s} \cos 2\pi s + 7^{-s} \cos 3\pi s + \cdots\right).$$

1. Let

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} = \int_0^1 \frac{1-x^{t-1}}{1-x}\,dx - \gamma,$$

where $t > 0$ and γ is the Eulerian constant; so that

(1) $$\psi(t+1) = \psi(t) + \frac{1}{t};$$

(2) $$\psi(1) = -\gamma$$

(3) and $$\lim_{t\to\infty}\{\psi(t) - \log t\} = 0.$$

Then we see that, if p, q and r are positive, then

(4) $$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{r x^{q-1}}{1-x^r}\right)dx = \lim_{s\to 0}\int_0^1 \left\{x^{p-1}(1-x)^{s-1} - r^{1-s}x^{q-1}(1-x^r)^{s-1}\right\}dx$$

$$= \lim_{s\to 0}\left\{\frac{\Gamma(p)\,\Gamma(s)}{\Gamma(p+s)} - r^{-s}\,\frac{\Gamma(q/r)\,\Gamma(s)}{\Gamma(s+q/r)}\right\}$$

$$= \frac{\Gamma'(q/r)}{\Gamma(q/r)} - \frac{\Gamma'(p)}{\Gamma(p)} + \log r$$

$$= \psi(q/r) - \psi(p) + \log r.$$

From this we deduce that

(5) $$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{b x^{bc-1}}{1-x^b}\right)\left(x^a + x^{ab} + x^{ab^2} + x^{ab^3} + \cdots + x^{ab^n}\right)dx$$

$$= \sum_{\nu=0}^{\nu=n}\left\{\psi(ab^{\nu-1}+c) - \psi(ab^{\nu}+c) + \log b\right\}$$

$$= \psi\left(\frac{a}{b}+c\right) - \psi(ab^n+c) + (n+1)\log b$$

provided that $a+c$, $a+bc$ and b are all positive. Supposing that $b > 1$ and making $n\to\infty$ we obtain from (5)

(6) $$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{b x^{bc-1}}{1-x^b}\right)\left(x^a + x^{ab} + x^{ab^2} + x^{ab^3} + \cdots\right)dx$$

$$= \psi\left(\frac{a}{b}+c\right) - \log\frac{a}{b}$$

provided that $b > 1$ and $a+c$ and $a+bc$ are positive. For example

Add. Ms a 7+ [18(28)] (25)

we have

$$(7) \qquad 1-\gamma = \int_0^1 \frac{x^2+x^4+x^8+x^{16}+\cdots}{1+x}\,dx,$$

$$(8) \qquad 1-\gamma = \int_0^1 \left(\frac{1+2x}{1+x+x^2}\right)(x^3+x^9+x^{27}+x^{81}+\cdots)\,dx,$$

$$(9) \qquad 1-\gamma = \int_0^1 \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})(1+\sqrt{x}+x)}\left\{x^{1/4}+x^{(1/4)^2}+x^{(1/4)^3}+\cdots\right\}dx,$$

and so on.

2. Similarly from the equation

$$(10) \qquad \int_0^1 \frac{x^{p-1}-x^{q-1}}{\log x}\,dx = \log\frac{q}{p}$$

which is true when p and q are positive, we can deduce that

$$(11) \qquad \int_0^1 \frac{x^{c-1}-x^{bc-1}}{\log x}(x^a+x^{ab}+x^{ab^2}+x^{ab^3}+\cdots)\,dx = \log\left(1+\frac{bc}{a}\right)$$

provided that $b>1$ and $a+c$ and $a+bc$ are positive.

3.

A Series for the Eulerian Constant — Add Mess 34 (18) (26) (21)

$$\gamma = \log 2 - \frac{2}{3^2-3} - 4\left(\frac{1}{6^2-6} + \frac{1}{9^2-9} + \frac{1}{12^2-12}\right)$$

$$- 6\left(\frac{1}{15^2-15} + \frac{1}{18^2-18} + \frac{1}{39^2-39}\right) - \&c$$

the last term in the nth group being $\frac{1}{\left(\frac{3n+3}{2}\right)^2 - \frac{3n+3}{2}}$

Now $\frac{1}{3n-1} + \frac{1}{3n} + \frac{1}{3n+1} = \frac{1}{n} + \frac{2}{(3n)^2-3n}$.

Putting $n = 1, 2, 3$ up to n and adding up all the results we have,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{3n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$+\left(\frac{2}{3^2-3} + \frac{2}{6^2-6} + \cdots \frac{2}{(3n)^2-3n}\right).$$

i.e $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots \frac{1}{3n+1}$

$$= 1 + \frac{2}{3^2-3} + \frac{2}{6^2-6} + \frac{2}{9^2-9} + \cdots + \frac{2}{(3n)^2-3n}.$$

Again putting $n = 1$, we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 1 + \frac{2}{3^2-3}$$

putting $n = 4$, we have

$$\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{13} = 1 + \frac{2}{3^2-3} + \frac{2}{6^2-6} + \frac{2}{9^2-9} + \frac{2}{12^2-12}$$

again when $n = 13$ we have

$$\frac{1}{14} + \frac{1}{15} + \cdots \frac{1}{40} = 1 + \frac{2}{3^2-3} + \cdots \frac{2}{39^2-39}$$

Continuing the process to n terms and adding up all the results we have

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\frac{3n-1}{2}}$$

$$= n + (n-1)\left(\frac{2}{3^2-3}\right) + (n-2)\left(\frac{2}{6^2-6} + \frac{2}{9^2-9} + \frac{2}{12^2-12}\right) +$$

$$(n-3)\left(\frac{2}{15^2-15} + \cdots + \frac{2}{39^2-39}\right) + \cdots \text{ to } n \text{ terms}.$$

But when n is very great we know that (27)

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \frac{1}{n-\frac{1}{2}}\right) - n \log 3$$

$$-\frac{1}{2n} + \log \frac{3n}{2n} = \gamma - \log 2 \quad \text{as } n \to \infty$$

Hence $\gamma = \log 2 - 2\left(\frac{1}{3^3-3}\right) + 4\left(\frac{1}{6^3-6} + \frac{1}{9^3-9} + \frac{1}{12^3-12}\right) - \&c$

$$\int_0^\infty x^{n-1} F(x)\, dx = \Gamma(n)\, u_{-n}$$

then $F(x) = u_0 - \frac{x}{1!} u_1 + \frac{x^2}{2!} u_2 - \cdots$

As $s \to 1$

$$\sum \frac{1}{n^s} = \frac{1}{s-1} + \gamma + o(1)$$

$$\sum \frac{\log n}{n^s} = \frac{1}{(s-1)^2} + K + o(1)$$

Hence $$\frac{\sum \frac{\log n}{n^s}}{\sum \frac{1}{n^s}} = \frac{1}{s-1} - \gamma + o(1)$$

But $$\frac{\sum \frac{\log n}{n^s}}{\sum \frac{1}{n^s}} = \sum -\frac{\log p}{p^s - 1}$$

$$\therefore \sum \frac{\log p}{p^s - 1} = \frac{1}{s-1} - \gamma + o(1)$$

(1) So that $$\sum \frac{1}{n^s} - \sum \frac{\log p}{p^s - 1} = 2\gamma + o(1)$$

(2) But $$\sum_{n<x} \frac{1}{n} = \log x + \gamma + o(1)$$

Assuming $\pi(x) \sim \frac{x}{\log x}$ we have

(3) $$\sum_{p<x} \frac{\log p}{p-1} = \log x + B + o(1).$$

Hence from (1), (2) and (3)

$$\gamma - B = 2\gamma$$

that is $$B = -\gamma$$

Hence we have

$$\frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \frac{\log 5}{5-1} + \frac{\log 7}{7-1} \cdots + \frac{\log p}{p-1} \quad p < x$$

$$= \log x - \gamma + o(1).$$

(1) $$\sum_{n<x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 - \gamma_1 + o(1)$$

where $\gamma_1 = .072815845483580$

(2) As $s \to 1$ $\quad \sum \frac{\log n}{n^s} = \frac{1}{(s-1)^2} - \gamma_1 + o(1)$

Since $\sum \frac{1}{n^s} = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + o(s-1)$

(3) $$\left(\sum \frac{1}{n^s}\right)^2 = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \gamma^2 + 2\gamma_1 + o(1)$$

Hence $= \sum \frac{d(n)}{n^s}$ where $d(n)$ is the no of divisors of n.

(4) But from (2) we have

$$\sum \frac{2\gamma + \log n}{n^s} = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + 2\gamma^2 - \gamma_1 + o(1)$$

Hence we have

(5) $$\sum \frac{2\gamma + \log n}{n^s} - \sum \frac{d(n)}{n^s}$$

$$= \gamma^2 - 3\gamma_1 + o(1)$$

But from (1) we have

(6) $$\sum_{n<x} \frac{2\gamma + \log n}{n} = \frac{1}{2}(\log x)^2 + 2\gamma \log x + 2\gamma^2 - \gamma_1 + o(1)$$

Assuming $\sum_{n<x} d(n) = x(2\gamma - 1 + \log x) + O(\sqrt{x})$

(7) $$\sum_{n<x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + 2\gamma \log x + B + o(1)$$

Hence from (5), (6) and (7), $2\gamma^2 - \gamma_1 - B = \gamma^2 - 3\gamma_1$

Hence $$\sum_{n<x} \frac{d(n)}{n} = (\gamma + \log x)^2 - \frac{1}{2}(\log x)^2 + 2\gamma_1 + o(1)..$$

* 2	* 7560	9292480	4900 8960
3	9240	9828800	5477 4720
* 4	* 10080	9977920	5654 8800
* 6	12600	10533360	6054 0480
8	13860	* 10810080	* 6126 1200
10	* 15120	13305630	6486 4800
* 12	18480	14137920	6846 8400
18	* 20160	* 14414400	* 7351 2440
20	* 25200	16633200	8216 2080
* 24	* 27720	18018900	8648 6400
30	30240	18894960	9189 1800
* 36	32760	19656600	9801 7920
* 48	36960	21067920	9945 9360
* 60	37800	* 21621600	10270 2600
72	40320	22827440	10744 2720
84	41580	* 22882880	10810 8000
90	42840	23226400	10954 9440
96	43680	* 23603600	* 11027 0160
108	* 45360	* 24324320	* 12255 2400
* 120	* 50400	25405400	13693 6800
168	* 55440	25654880	* 14702 6880
* 180	65520	25765760	16432 4160
* 240	75600	26126120	* 19378 3600
336	* 83160	26320160	20540 5200
* 360	98280	* 26486480	22054 0320
420	* 110880	* 27207200	23227 9260
480	131040	* 28648640	* 24504 4800
504	138600	* 29810800	27387 3600
540	150840	32252240	* 29406 3760
600	163800	32972760	32864 8320
630	* 166320	36936880	34918 8840
660	196560	41427200	* 36756 7200
672	* 221760	* 44114400	41081 0400
* 720	262080	* 47297280	46558 5120
* 840	* 277200	48378760	49008 9600
1080	327600	50540520	49729 6800
* 1260	* 332640	* 52162600	51459 4080
1440	360360	54504480	53721 3600
* 1680	393120	57387360	54774 7200
2160	443520	58274400	* 55135 0800
* 2520	471240	58828800	61621 5600
3360	480480	60270240	* 69837 7680
3780	* 498760	60630600	* 72513 4400
3960	* 554400	63600800	82162 0800
4320	655200	* 64232400	93117 0240
4620	* 665280	* 64756720	99459 3600
4680	* 720720	64981040	102918 8160
* 5040	831600	* 43243200	107442 7200

Add the q_n (30)

Continuation of "Highly Composite Numbers" 15 p. 78.

Add the q_n (67)

them is $\nu_1 \geq 0$. Then it can be shown that

$$(268) \quad \left(1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots\right)^2$$

$$= 1 + 4\left(\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \frac{q^7}{1-q^7} + \cdots\right)$$

$$= 1 + 4\left\{Q_2(1)q + Q_2(2)q^2 + Q_2(3)q^3 + \cdots\right\}.$$

From this it easily follows that

$$(269) \quad \zeta(s)\zeta_1(s) = \frac{Q_2(1)}{1^s} + \frac{Q_2(2)}{2^s} + \frac{Q_2(3)}{3^s} + \cdots,$$

where

$$\zeta_1(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \cdots.$$

Since

$$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \cdots$$

$$= d(1)q + d(2)q^2 + d(3)q^3 + \cdots,$$

it follows from (268) that

$$(270) \quad Q_2(N) \leqq d(N)$$

for all values of N. Let

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p},$$

where $a_\lambda \geq 0$. Then we see that, if any one of $a_3, a_7, a_{11}, \cdots$ be odd, where 3, 7, 11, ... are the primes of the form $4n-1$, then

$$(271) \quad Q_2(N) = 0.$$

But, if $a_3, a_7, a_{11} \cdots$ be even or zero, then

$$(272) \quad Q_2(N) = (1 + a_5)(1 + a_{13})(1 + a_{17}) \cdots,$$

where 5, 13, 17 ... are the primes of the form $4n+1$.

It is clear that (270) is a consequence of (271) and (272).

53. From (272) it is easy to see that, in order that $Q_2(N)$

with Add MS a 94[18] (31)

Apparently not the same as 19. p 163.

Proof of statement made at beginning is
completed

in the second series

We can then differentiate the series term by term

$$\frac{d}{x} E(x)$$

[illegible] on the reverse side of
[illegible] note with Add MS a 94, 18(31)

And if $a_4 \ldots$ (68)

should be of maximum order, N must be of the form

$$5^{a_5} \cdot 13^{a_{13}} \cdot 17^{a_{17}} \cdot \ldots \cdot p^{a_p},$$

where p is a prime of the form $4n+1$, and

$$a_5 \geqslant a_{13} \geqslant a_{17} \geqslant \ldots \geqslant a_p.$$

Let $\pi_1(x)$ denote the number of primes of the form $4n+1$ which do not exceed x, and let

$$\vartheta_1(x) = \log 5 + \log 13 + \log 17 + \ldots + \log p,$$

where p is the largest prime of the form $4n+1$, not greater than x. Then by arguments similar to those of § 33 we can show that

$$(27.3) \quad Q_2(N) \leqslant N^{\frac{1}{x}} \frac{2^{\pi_1(2^x)}}{e^{\frac{1}{x}\vartheta_1(2^x)}} \frac{\left(\frac{3}{2}\right)^{\pi_1\left(\frac{3}{2}\right)^x}}{e^{\frac{1}{x}\vartheta_1\left(\frac{3}{2}\right)^x}} \frac{\left(\frac{4}{3}\right)^{\pi_1\left(\frac{4}{3}\right)^x}}{e^{\frac{1}{x}\vartheta_1\left(\frac{4}{3}\right)^x}} \ldots$$

for all values of N and x. From this we can show by arguments similar to those of § 38 that, in order that $Q_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_1(2^x) + \vartheta_1\left(\frac{3}{2}\right)^x + \vartheta_1\left(\frac{4}{3}\right)^x + \ldots},$$

and $Q_2(N)$ of the form

$$2^{\pi_1(2^x)} \left(\frac{3}{2}\right)^{\pi_1\left(\frac{3}{2}\right)^x} \left(\frac{4}{3}\right)^{\pi_1\left(\frac{4}{3}\right)^x} \ldots$$

Then, without assuming the prime number theorem, we can show that the maximum order of $Q_2(N)$ is

$$(27.4) \quad 2^{\log N \left\{ \frac{1}{\log\log N} + O\left(\frac{1}{(\log\log N)^2}\right) \right\}}.$$

Assuming the prime number theorem we can show that the maximum order of $Q_2(N)$ is

$$(27.5) \quad 2^{\frac{1}{2} \mathrm{Li}(2 \log N) + O\left\{\log N \, e^{-a\sqrt{(\log N)}}\right\}},$$

where a is a positive constant.

54. We shall now assume the Riemann Hypothesis and its analogue for the function $\zeta_1(s)$. Let ρ_1 be a complex root of $\zeta_1(s)$. Then it can be shown that

$$\sum \frac{1}{\rho_1} = \frac{\gamma - 3\log\pi}{2} + \log 2 + 4\log\Gamma\left(\tfrac{3}{4}\right);$$

so that

$$\sum \frac{1}{\rho} + \sum \frac{1}{\rho_1} = 1 + \gamma - 2\log\pi + 4\log\Gamma\left(\tfrac{3}{4}\right). \tag{53.0}$$

It can also be shown that

$$\begin{cases} 2\vartheta_1(x) = x - 2\sqrt{x} - \sum x^{\rho} - \sum x^{\rho_1} + O(x^{\frac{1}{3}}) \\ 2\pi_1(x) = Li(x) - Li(\sqrt{x}) - \sum Li(x^{\rho}) - \sum Li(x^{\rho_1}) + O(x^{\frac{1}{3}}), \end{cases} \tag{53.1}$$

so that

$$\begin{cases} 2\vartheta_1(x) = x + O\sqrt{x}(\log x)^2 \\ 2\pi_1(x) = Li(x) + O\sqrt{x}\log x. \end{cases} \tag{53.2}$$

Now

$$2\pi_1(x) = Li(x) - \frac{1}{\log x}\left(2\sqrt{x} + \sum x^{\rho} + \sum x^{\rho_1}\right) - \frac{1}{(\log x)^2}\left(4\sqrt{x} + \sum x^{\rho} + \sum x^{\rho_1}\right) + O\frac{\sqrt{x}}{(\log x)^3}.$$

But by Taylor's Theorem we have

$$Li\{2\vartheta_1(x)\} = Li(x) - \frac{1}{\log x}\left(2\sqrt{x} + \sum x^{\rho} + \sum x^{\rho_1}\right) + O(\log x)^2.$$

Hence

$$2\pi_1(x) = Li\{2\vartheta_1(x)\} - 2R_1(x) + O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\}, \tag{53.1}$$

where

$$R_1(x) = \frac{1}{(\log x)^2}\left(2\sqrt{x} + \tfrac{1}{2}\sum x^{\rho} + \tfrac{1}{2}\sum x^{\rho_1}\right).$$

It can easily be shown that

$$\sqrt{x}\left(2 + \sum \frac{1}{\rho} + \sum \frac{1}{\rho_1}\right) \geq R_1(x)(\log x)^2 \geq \sqrt{x}\left(2 - \sum \frac{1}{\rho} - \sum \frac{1}{\rho_1}\right);$$

and so from (276) we see that

(70)

(280) $\left\{ 3+\gamma - 2\log\pi + 4\log\Gamma(\tfrac{3}{4}) \right\}\sqrt{x} \fallingdotseq R_1(x)(\log x)^2 \fallingdotseq \left\{ 1-\gamma+2\log\pi - 4\log\Gamma(\tfrac{3}{4}) \right\}\sqrt{x}$

can easily be verified that

(281) $$\begin{cases} 3+\gamma-2\log\pi+4\log\Gamma(\tfrac{3}{4}) = 2.101 \\ 1-\gamma+2\log\pi-4\log\Gamma(\tfrac{3}{4}) = 1.899, \end{cases}$$

approximately.

Proceeding as in §43 we can show that the maximum order of $Q_2(N)$ is

(283) $$2^{\frac{1}{2}} \mathrm{Li}(2\log N) + \phi(N),$$

where

$$\phi(N) = \frac{\log(\frac{3}{2})}{2\log 2}\,\mathrm{Li}\left\{ \tfrac{3}{2}(\log N)^{\frac{\log(3/2)}{\log 2}} \right\} - \frac{3}{4}(\log N)^{\frac{\log(3/2)}{\log 2}} - \frac{3}{4\log(2\log N)} - R_1(2\log N) + O\left\{ \frac{\sqrt{(\log N)}}{(\log\log N)^2} \right\}$$

15. Let $\overline{Q}_2(N)$ denote the number of ways in which N can be expressed as m^2+mn+n^2. Let us agree to consider m^2+mn+n^2 as two ways if m and n are unequal, and as one way if they are equal or one of them is zero. Then it can be shown that

(285) $$\begin{aligned} &\tfrac{1}{2}(1+2q^{\frac{1}{4}}+2q^{\frac{4}{4}}+2q^{\frac{9}{4}}+\cdots)(1+2q^{\frac{3}{4}}+2q^{\frac{12}{4}}+2q^{\frac{27}{4}}+\cdots) \\ &+\tfrac{1}{2}(1-2q^{\frac{1}{4}}+2q^{\frac{4}{4}}-2q^{\frac{9}{4}}+\cdots)(1-2q^{\frac{3}{4}}+2q^{\frac{12}{4}}-2q^{\frac{27}{4}}+\cdots) \\ &= 1+6\left(\frac{q}{1-q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^5}{1-q^5}+\cdots\right) \\ &= 1+6\left\{ \overline{Q}_2(1)q+\overline{Q}_2(2)q^2+\overline{Q}_2(3)q^3+\cdots \right\}, \end{aligned}$$

where 1, 2, 4, 5, ... are the natural numbers without the multiples of 3. From this it follows that

(286) $$\zeta(s)\,\zeta_2(s) = 1^{-s}\overline{Q}_2(1)+2^{-s}\overline{Q}_2(2)+3^{-s}\overline{Q}_2(3)+\cdots\cdots$$

where

$$\zeta_2(s) = 1^{-s} - 2^{-s} + 4^{-s} - 5^{-s} + \dots \dots$$

It also follows that

(2x.1) $$\overline{Q}_2(N) \leq d(N),$$

for all values of N. Let

$$N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \dots p^{a_p},$$

where $a_2 \geq 2 \geq 0$. Then, if any one of $a_2, a_5, a_{11}, \dots$ be odd, the where $2, 5, 11, \dots$ are primes of the form $3n-1$, then

(2x.2) $$\overline{Q}_2(N) = 0.$$

But, if $a_2, a_5, a_{11}, \dots$ be even or zero, then

(2x.3) $$\overline{Q}_2(N) = (1+a_7)(1+a_{13})(1+a_{19})(1+a_{31}) \dots,$$

where $7, 13, 19, \dots$ are the primes of the form $6n+1$.

Let $\pi_2(x)$ be the number of primes of the form $6n+1$ which do not exceed x, and let

$$\vartheta_2(x) = \log 7 + \log 13 + \log 19 + \dots + \log p,$$

where p is the largest prime of the form $6n+1$ not greater than x. Then we can show that, in order that $\overline{Q}_2(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta_2(2^x) + \vartheta_2(\frac{3}{2})^x + \vartheta_2(\frac{4}{3})^x + \dots}$$

and $\overline{Q}_2(N)$ of the form

$$2^{\pi_2(3^x)} \left(\tfrac{3}{2}\right)^{\pi_2(\frac{3}{2})^x} \left(\tfrac{4}{3}\right)^{\pi_2(\frac{4}{3})^x} \dots \dots.$$

Without assuming the prime number theorem we can show that the maximum order of $\overline{Q}_2(N)$ is

(72)

(2·8·8) $$2^{\log N}\left\{\frac{1}{\log\log N}+O\frac{1}{(\log\log N)^2}\right\}$$

Assuming the prime number theorem we can show that the maximum order of $R_0(N)$ is

(2·8·7) $$2^{\frac{1}{2}\mathrm{Li}(2\log N)+O\{\log N e^{-a\sqrt{(\log N)}}\}}$$

56. We shall now assume the Riemann hypothesis and its analogue for the function $\zeta_2(s)$. Then we can show that

(2·10) $$2\pi_2(x)=\mathrm{Li}\{2\vartheta_1(x)\}-2R_2(x)+O\left\{\frac{\sqrt{x}}{(\log x)^3}\right\},$$

where $$R_2(x)=\frac{1}{(\log x)^2}\left\{2\sqrt{x}+\tfrac{1}{2}\sum\frac{x^{\rho}}{\rho}+\tfrac{1}{2}\sum\frac{x^{\rho_2}}{\rho_2}\right\},$$

where ρ_2 is a complex root of $\zeta_2(s)$. It can also be shown that

(2·11) $$\sum\frac{1}{p}+\sum\frac{1}{\rho_2}=1+\gamma+\tfrac{1}{2}\log 3+3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})};$$

and so

(2·12) $$\left\{3+\gamma+\tfrac{1}{2}\log 3+3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}\sqrt{x}\geqslant R_2(x)(\log x)^2\geqslant\left\{1-\gamma-\tfrac{1}{2}\log 3-3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\right\}\sqrt{x}$$

It can easily be verified that

(2·13) $$\begin{cases}3+\gamma+\frac{1}{2}\log 3+3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}=2\cdot080,\\ 1-\gamma-\frac{1}{2}\log 3-3\log\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}=1\cdot920,\end{cases}$$

approximately. Then we can show that the maximum order of $\overline{R}_2(N)$ is

(2·14) $$2^{\frac{1}{2}\mathrm{Li}(2\log N)+\phi(N)}$$

where $$\phi(N)=\frac{\log(3/2)}{\frac{1}{2}\log 2}\mathrm{Li}\left\{\tfrac{3}{2}(\log N)^{\frac{\log(3/2)}{\log 2}}\right\}-\frac{3(\log N)^{\frac{\log(3/2)}{\log 2}}}{4\log(2\log N)}$$
$$-R_2(\log N)+O\left\{\frac{\sqrt{(\log N)}}{(\log\log N)^3}\right\}.$$

(73)

57. Let $d_n(N)$ denote the coefficient of N^{-s} in the expansion of $\{\zeta(s)\}^{1+n}$ as a Dirichlet series. Then since

$$\{\zeta(s)\}^{-1} = (1-2^{-s})(1-3^{-s})(1-5^{-s}) \cdots (1-p^{-s}) \cdots,$$

it is easy to see that, if

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n},$$

where $p_1, p_2, p_3 \cdots$ are any primes, then

$$d_n(N) = \prod_{\nu=1}^{\nu=n} \prod_{\lambda=1}^{\lambda=a_\nu} \left(1+\frac{n}{\lambda}\right). \tag{295}$$

provided that $n > -1$. It is evident that

$$d_{-1}(N) = 0, \quad d_0(N) = 1, \quad d_1(N) = d(N);$$

and that, if $-1 \leqq n \leqq 0$, then

$$d_n(N) \leqq 1+n \tag{296}$$

for all values of N. It is also evident that, if N is a prime then

$$d_n(N) = 1+n$$

for all values of n. It is easy to see from (295) that, if $n>0$ then $d_n(N)$ is not bounded when N becomes infinite. Now, if n is positive, it can easily be shown that, in order that $d_n(N)$ should be of maximum order, N must be of the form

$$e^{\vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots},$$

and consequently $d_n(N)$ of the form

$$(1+n)^{\pi(x_1)} \left(1+\frac{n}{2}\right)^{\pi(x_2)} \left(1+\frac{n}{3}\right)^{\pi(x_3)} \cdots;$$

and proceeding as in §46 we can show that N must be of the form

$$e^{\vartheta(1+p_1^{-1})^x + \vartheta(1+\frac{1}{2})^x + \vartheta(1+\frac{1}{3})^x + \cdots} \tag{297}$$

and $d_x(N)$ of the form

$$(1+\lambda)^{\pi(1+\lambda)^x}\left(1+\tfrac{2}{3}\right)^{\pi(1+\frac{2}{3})^x}\left(1+\tfrac{2}{5}\right)^{\pi(1+\frac{2}{5})^x}\cdots\cdots \tag{298}$$

From (297) and (298) we can easily find the maximum order of $d_x(N)$ as in §43.

It may be entertaining to note that numbers of the form (297) which may also be written in the form

$$e^{x\vartheta\left\{x\frac{1}{x}\log(1+x)\right\}}+e^{x\vartheta\left\{x\frac{1}{x}\log(1+\frac{2}{3})\right\}}+e^{x\vartheta\left\{x\frac{1}{x}\log(1+\frac{2}{5})\right\}}+\cdots.$$

approach the form

$$e^{x\vartheta(x)}+e^{\vartheta(\sqrt{x})}+e^{\vartheta(x^{1/3})}+\cdots.$$

as $x\to\infty$. That is to say, they approach the form of the least common multiple of the natural numbers as $x\to\infty$.

(3·1) $$\sigma_{-s}(N) < \frac{\left\{1-(p_1 p_2 p_3 \cdots p_n N)^{-s/n}\right\}^n}{(1-p_1^{-1})(1-p_2^{-s})\cdots\cdots(1-p_n^{-s})}.$$

By arguments similar to those of § 2 we can show that it is possible to choose the indices $a_1, a_2, a_3 \cdots a_n$ so that

(3·2) $$\sigma_{-s}(N) = \frac{\left\{1-(p_1 p_2 p_3 \cdots p_n N)^{-s/n}\right\}^n}{(1-p_1^{-1})(1-p_2^{-s})\cdots\cdots(1-p_n^{-s})}\left\{1-O\left\{N^{-s/n}(\log N)^{-2/(n-1)}\right\}\right\}.$$

There are of course results corresponding to (1·4) and (1·5) also.

59. N may be said to be a generalised highly composite number (number) if $\sigma_{-s}(N) > \sigma_{-s}(N')$ for all values of N' less than N. We can easily show that, in order that N should be a generalised highly composite number, N must be of the form

(3·3) $$2^{a_2} 3^{a_3} 5^{a_5} \cdots\cdots p^{a_p},$$

where $a_2 \geqslant a_3 \geqslant a_5 \geqslant \cdots\cdots \geqslant a_p = 1$,

the exceptional numbers being 36, for the values of s which satisfy the inequality $2^s+4^s+8^s > 3^s+9^s$ and 4 in all cases.

A number N may be said to be a generalised superior highly composite number if there is a positive number ε such that

(3·4) $$\frac{\sigma_{-s}(N)}{N^\varepsilon} \geqslant \frac{\sigma_{-s}(N')}{(N')^\varepsilon},$$

for all values of N' less than N, and

(3·5) $$\frac{\sigma_{-s}(N)}{N^\varepsilon} > \frac{\sigma_{-s}(N')}{(N')^\varepsilon}$$

for all values of N' greater than N. It is easily seen that all generalised superior highly composite numbers are also generalised highly composite numbers. We shall use the expression

$$2^{a_2} 3^{a_3} 5^{a_5} \cdots\cdots p^{a_p},$$

and the expression

(76)

$$\begin{array}{l} 2.3.5.7\cdots p_1 \\ \times 2.3.5.\cdots p_2 \\ \times 2.3.5.\cdots p_3 \\ \times \cdots \\ \cdots \\ \cdots \end{array}$$

as the standard forms of a generalised superior highly composite number.

6. Let $$N' = \frac{N}{\lambda},$$

where $\lambda \leq p_1$. Then from (304) it follows that

$$1 - \lambda^{-s(1+a_\lambda)} \geqslant (1 - \lambda^{-sa_\lambda})\,\lambda^{\varepsilon},$$

(306) or $$\lambda^{\varepsilon} \leqslant \frac{1 - \lambda^{-s(1+a_\lambda)}}{1 - \lambda^{-sa_\lambda}}$$

Again let $N' = N\lambda$. Then from (305) we see that

$$1 - \lambda^{-s(1+a_\lambda)} > \{1 - \lambda^{-s(2+a_\lambda)}\}\,\lambda^{-\varepsilon}$$

(307) or $$\lambda^{\varepsilon} > \frac{1 - \lambda^{-s(2+a_\lambda)}}{1 - \lambda^{-s(1+a_\lambda)}}.$$

Now let us suppose that $\lambda = p_1$ in (306) and $\lambda = P_1$ in (307). Then we see that

(308) $$\frac{\log(1 + p_1^{-s})}{\log p_1} \geqslant \varepsilon > \frac{\log(1 + P_1^{-s})}{\log P_1}.$$

From this it follows that, if

$$0 < \varepsilon \leq \frac{\log(1 + 2^{-s})}{\log 2},$$

then there is a unique value of p_1 corresponding to each value of ε. It follows from (306) that

(309) $$a_\lambda \leqslant \frac{\log\left(\frac{\lambda^{\varepsilon} - \lambda^{-s}}{\lambda^{\varepsilon} - 1}\right)}{s \log \lambda},$$

and from (307) that

(310) $$1 + a_\lambda > \frac{\log\left(\frac{\lambda^{\varepsilon} - \lambda^{-s}}{\lambda^{\varepsilon} - 1}\right)}{s \log \lambda}.$$

From (309) and (310) it is clear that

$$a_\lambda = \left[\frac{\log\left(\frac{\lambda^\varepsilon - \lambda^{-s}}{\lambda^\varepsilon - 1}\right)}{s \log \lambda} \right] \tag{311}$$

Hence N is of the form

$$2^{\left[\frac{\log\left(\frac{2^\varepsilon - 2^{-s}}{2^\varepsilon - 1}\right)}{s \log 2}\right]} \; 3^{\left[\frac{\log\left(\frac{3^\varepsilon - 3^{-s}}{3^\varepsilon - 1}\right)}{s \log 3}\right]} \cdots \cdots \cdots p_1^{\cdots}, \tag{312}$$

where p_1 is the prime defined by the inequalities (308).

61. Let us consider the nature of p_n. Putting $\lambda = p_n$ in (306), and remembering that $a_{p_n} \geqslant n$, we obtain

$$p_n^\varepsilon \leqslant \frac{1 - p_n^{-s(1+a_{p_n})}}{1 - p_n^{-s}\, a_{p_n}} \leqslant \frac{1 - p_n^{-s(n+1)}}{1 - p_n^{-sn}}. \tag{313}$$

Again, putting $\lambda = p_n$ in (307), and remembering that $a_{p_n} \leqslant n-1$, we obtain

$$p_n^\varepsilon > \frac{1 - p_n^{-s(n+a_{p_n})}}{1 - p_n^{-s(1+a_{p_n})}} \geqslant \frac{1 - p_n^{-s(n+1)}}{1 - p_n^{-sn}} \tag{314}$$

It follows from (313) and (314) that, if x_n be the value of x satisfying the equation

$$x^\varepsilon = \frac{1 - x^{-s(n+1)}}{1 - x^{-sn}}, \tag{315}$$

then p_n is the largest prime not greater than x_n. Hence N is of the form

$$e^{\theta(x_1) + \theta(x_2) + \theta(x_3) + \cdots} \tag{316}$$

where x_n is defined in (315); and $\sigma_{-s}(N)$ is of the form

$$\Pi_1(x_1)\, \Pi_2(x_2)\, \Pi_3(x_3) \cdots \cdots \Pi_{k_2}(x_{a_2}), \tag{317}$$

where

$$\Pi_n(x) = \frac{1 - 2^{-s(n+1)}}{1 - 2^{-sn}} \cdot \frac{1 - 3^{-s(n+1)}}{1 - 3^{-sn}} \cdots \frac{1 - p^{-s(n+1)}}{1 - p^{-sn}},$$

and p is the largest prime not greater than x.

62. It follows from (304) and (305) that

$$\sigma_{-s}(N) \leqslant N^\varepsilon \frac{\Pi_1(x_1)}{e^{\varepsilon\, \theta(x_1)}} \frac{\Pi_2(x_2)}{e^{\varepsilon\, \theta(x_2)}} \frac{\Pi_3(x_3)}{e^{\varepsilon\, \theta(x_3)}} \cdots \tag{318}$$

for all values of N, where $x_1, x_2, x_3, \ldots$ are functions of ε

defined by the equations (78)

(319) $$x_n^{\varepsilon} = \frac{1 - x_n^{-s(n+1)}}{1 - x_n^{-sn}} ;$$

and $\sigma_{-s}(N)$ is equal to the right-hand side of (318) when

$$N = e^{\vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots}$$

69. In (16) let us suppose that

$$\phi(x) = \log \frac{1 - x^{-s(n+1)}}{1 - x^{-sn}} .$$

Then we see that

$$\begin{aligned} \log \Pi_n(x_n) &= \pi(x_n) \log \frac{1 - x_n^{-s(n+1)}}{1 - x_n^{-sn}} - \int \pi(x_n)\, d\left(\log \frac{1 - x_n^{-s(n+1)}}{1 - x_n^{-sn}}\right) \\ &= \pi(x_n) \log(x_n^{\varepsilon}) - \int \pi(x_n)\, d(\log x_n^{\varepsilon}) \\ &= \varepsilon\, \pi(x_n) \log x_n - \int \pi(x_n) \log x_n\, d\varepsilon - \int \frac{\varepsilon\, \pi(x_n)}{x_n}\, dx_n \end{aligned}$$

in virtue of (319). Hence

(320) $$\begin{aligned} \log \Pi_n(x_n) - \varepsilon\, \vartheta(x_n) &= \varepsilon \left\{ \pi(x_n) \log x_n - \vartheta(x_n) \right\} - \int \pi(x_n) \log x_n\, d\varepsilon - \int \frac{\varepsilon\, \pi(x_n)}{x_n}\, dx_n \\ &= \varepsilon \int \frac{\pi(x_n)}{x_n}\, dx_n - \int \pi(x_n) \log x_n\, d\varepsilon - \int \frac{\varepsilon\, \pi(x_n)}{x_n}\, dx_n \\ &= \int d\varepsilon \int \frac{\pi(x_n)}{x_n}\, dx_n - \int \pi(x_n) \log x_n\, d\varepsilon \\ &= \int \left\{ \int \frac{\pi(x_n)}{x_n}\, dx_n - \pi(x_n) \log x_n \right\} d\varepsilon \\ &= - \int \vartheta(x_n)\, d\varepsilon \end{aligned}$$

It follows from (318) and (320) that

(321) $$\sigma_{-s}(N) \leq N^{\varepsilon} e^{-\int \{ \vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots \} d\varepsilon}$$

for all values of N. By arguments similar to those of §38 we can show that the right-hand side of (321) is a minimum when ε is a function of N defined by the equation

(322) $$N = e^{\vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots} .$$

Now let $\sum_{-s}(N)$ be a function of N defined by the equation

(323) $$\sum_{-s}(N) = \Pi_1(x_1)\, \Pi_2(x_2)\, \Pi_3(x_3) \cdots$$

[Middle of another paper]

* * * * * * (79).

where ϵ is a function of N defined by the equation (322). Then it follows from (318) that the order of

$$\overline{\sigma}_{-s}(N) \leqq \overline{\Sigma}_{-s}(N)$$

for all values of N and $\overline{\sigma}_{-s}(N) = \overline{\Sigma}_{-s}(N)$ for all generalized superior highly composite numbers values of N. In other words $\overline{\sigma}_{-s}(N)$ is of maximum order when N is of the form of a generalized superior highly composite number.

63. We shall now consider some important series which are not only useful in finding the maximum order of $\overline{\sigma}_{-s}(N)$ but also interesting in themselves. Proceeding as in (16) we can easily show that, if $\phi'(x)$ be continuous, then

$$\begin{aligned}(324)\quad \phi(2)\log 2 + \phi(3)\log 3 + \phi(5)\log 5 + \cdots + \phi(p)\log p \\ = \phi(x)\,\vartheta(x) - \int_a^x \phi'(t)\,\vartheta(t)\,dt.\end{aligned}$$

where p is the largest prime not exceeding x.

Since $\int \phi(x)\,dx = x\,\phi(x) - \int x\,\phi'(x)\,dx$, we have

$$\begin{aligned}(325)\quad \phi(x)\,\vartheta(x) - \int \phi'(x)\,\vartheta(x)\,dx \\ = \int \phi(x)\,dx - \{x - \vartheta(x)\}\,\phi(x) + \int \phi'(x)\,\{x - \vartheta(x)\}\,dx.\end{aligned}$$

Remembering that $x - \vartheta(x) = O\{\sqrt{x}\,(\log x)^2\}$, we have by Taylor's theorem

$$(326)\quad \int^{\vartheta(x)} \phi(t)\,dt = \int \phi(x)\,dx - \{x - \vartheta(x)\}\,\phi(x) + \tfrac{1}{2}\{x - \vartheta(x)\}^2\,\phi'\{x + O(\sqrt{x}(\log x)^2)\}$$

It follows from (324) - (326) that

$$\begin{aligned}(327)\quad \phi(2)\log 2 + \phi(3)\log 3 + \phi(5)\log 5 + \cdots + \phi(p)\log p \\ = C + \int^{\vartheta(x)} \phi(t)\,dt + \int \phi'(x)\,\{x - \vartheta(x)\}\,dx - \tfrac{1}{2}\{x - \vartheta(x)\}^2\,\phi'\{x + O(\sqrt{x}(\log x)^2)\}\end{aligned}$$

where C is a constant and p is the largest prime not exceeding x. (80)

64. Now let us assume that $\phi(x) = \frac{1}{x^s - 1}$ where $s > 0$. Then from (327) we see that, if p is the largest prime not greater than x, then

$$\begin{aligned}(328)\quad \frac{\log 2}{2^s - 1} + \frac{\log 3}{3^s - 1} + \frac{\log 5}{5^s - 1} + \cdots + \frac{\log p}{p^s - 1} \\ = C + \int^{\vartheta(x)} \frac{dx}{x^s - 1} - s\int \frac{x - \vartheta(x)}{x^{s+1}(x^s - 1)^2}\,dx + O\{x^{-s}(\log x)^2\}\end{aligned}$$

But it is known that

$$(329)\quad x - \vartheta(x) = \sqrt{x} + x^{\frac{1}{3}} + \sum \frac{x^\rho}{\rho} - \Sigma\frac{x^{\frac{1}{2}\rho}}{\rho} + O(x^{\frac{1}{4}})$$

where ρ is a complex root of $\zeta(s)$. By arguments similar to those of §42 we can show that

$$\sum \frac{x^{\frac{1}{2}\rho - s}}{\rho(\frac{1}{2}\rho - s)} = \int x^{-1-s} \sum \frac{x^{\frac{1}{2}\rho}}{\rho}\,dx$$

Hence $\int_{\cdot}^{\infty} \frac{\sum \frac{x^{\frac{1}{\rho}}}{\rho}}{x^{\frac{1}{2}}(x^2-1)}\,dx = \int O\left\{x^{-\frac{1}{2}}\sum \frac{x^{\frac{1}{\rho}}}{\rho}\right\}dx$

$= O\left\{\sum \frac{x^{\frac{1}{\rho}-\frac{1}{2}}}{\rho(\frac{1}{\rho}-\frac{1}{2})}\right\} = O(x^{\frac{1}{2}-\varepsilon})$

Similarly

$\int \frac{\sum \frac{x^{\rho}}{\rho}}{x^{\frac{1}{2}}(x^2-1)}\,dx = \sum \frac{x^{\rho-\frac{1}{2}}}{\rho(\rho-\frac{1}{2})} + O\left(\sum \frac{x^{\rho-\frac{1}{2}}}{\rho(\rho-\frac{1}{2}+1)}\right) = \sum \frac{x^{\rho-\frac{1}{2}}}{\rho(\rho-\frac{1}{2})} + O(x^{\frac{1}{2}-1+\varepsilon})$

Hence (328) may be replaced by

(330) $\frac{\log 2}{2^{\frac{1}{2}}-1} + \frac{\log 3}{3^{\frac{1}{2}}-1} + \frac{\log 5}{5^{\frac{1}{2}}-1} + \cdots + \frac{\log n}{n^{\frac{1}{2}}-1}$

$= C + \int^{\theta(x)} \frac{dt}{t^{\frac{1}{2}}-1} - \frac{1}{2}\int \frac{x^{\frac{1}{2}}+x^{\frac{1}{2}}}{x^{\frac{1}{2}+1}(x^2-1)^2}\,dx - \frac{1}{2}\sum \frac{x^{\rho-\frac{1}{2}}}{\rho(\rho-\frac{1}{2})} + O\left(x^{\frac{1}{2}-\frac{1}{2}\theta} + x^{\frac{1}{2}-\varepsilon}\right)$

It can easily be shown that

(331) $C = -\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})}$.

when the error term is $\Re\, o(1)$

65. Let $S_s(x) = -s\sum \frac{x^{\rho-s}}{\rho(\rho-s)}$ (61)

Then

(332) $|S_s(x)| \leqslant s\sum \left|\frac{x^{\rho-s}}{\rho(\rho-s)}\right| = s.x^{\frac{1}{2}-\sigma}\sum \frac{1}{\sqrt{|\rho(1-\rho)(\rho-s)(1-\rho-s)|}}$.

If m and n are any two positive numbers, then it is evident that $1/\sqrt{mn}$ lies between $\frac{1}{m}$ and $\frac{1}{n}$.

Hence $\sum \frac{1}{\sqrt{|\rho(1-\rho)(\rho-s)(1-\rho-s)|}}$ lies between $\chi(s)$ and $\chi(s)$ where

(333) $\chi(s) = \sum \frac{1}{(\rho-s)(1-\rho-s)} = \sum \frac{1}{\rho(1-\rho)+s^2-s} = \frac{1}{1-2s}\left(\sum \frac{1}{\rho-s} + \sum \frac{1}{1-\rho-s}\right)$

$= \sum \frac{\frac{1}{2}s}{\frac{1}{4}s^2-\frac{1}{4}}$

We can show as in § 41 that

(334) $\sum \frac{1}{s-\rho} = \frac{2s-1}{s^2-s} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} + \frac{\zeta'(s)}{\zeta(s)}$

Hence

(335) $\chi(s) = \frac{2}{s^2-s} + \frac{1}{2s-1}\left\{\frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} + 2\frac{\zeta'(s)}{\zeta(s)} - \log\pi\right\}$

so that (336) $\chi(0) = \chi(1) = 2 - 2 + \gamma - \log 4\pi$.

By elementary algebra, it can easily be shown that if m_1 and n_1 be not negative and G_1 be the geometric mean between m_1 and n_1 then

(337) $G_1 + G_2 + G_3 + \cdots < \sqrt{\{(m_1+n_1+m_2+n_2+\cdots)\}}$

unless $\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3} = \cdots$.

From this it follows that

(338) $\sum \frac{1}{\sqrt{|\rho(1-\rho)(\rho-s)(1-\rho-s)|}} < \sqrt{|\chi(1)\chi(s)|}$.

The following method leads to still closer approximations. It is easy to see that if m and n are positive, then $1/\sqrt{mn}$ is the geometric mean between

(339) $\frac{1}{2m} + \frac{8}{2(m+2n)}$ and $\frac{1}{2n} + \frac{8}{2(2m+n)}$;

and so $\frac{1}{\sqrt{mn}}$ lies between both. Hence (340)

$\sum \frac{1}{\sqrt{|\rho(1-\rho)(\rho-s)(1-\rho-s)|}}$ lies between $\frac{1}{2}\sum \frac{1}{\rho(1-\rho)} + \frac{1}{2}\sum \frac{1}{\rho(1-\rho)+\frac{2}{9}(s^2-s)}$ and $\frac{1}{2}\sum \frac{1}{(\rho-s)(1-\rho-s)} + \frac{1}{2}\sum \frac{1}{\rho(1-\rho)+\frac{2}{9}(s^2-s)}$

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and is also less than the Geometric mean* between them for $\rho > 1$ (since $\rho(1-\rho) < 0$ for $\rho > 1$)

If (322) $\frac{1}{\sqrt{\rho(1-\rho)(\rho-\gamma)(1-\rho-\gamma)}} < \frac{1}{\rho(1-\rho)} - \frac{1}{2}\frac{\gamma^2-\gamma}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^2 - \frac{5}{16}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^3 \cdots$

for

$$\frac{1}{3}\frac{1}{\rho(1-\rho)} + \frac{2}{3}\frac{1}{\rho(1-\rho)+\frac{2}{9}(\gamma^2-\gamma)} < \frac{1}{\rho(1-\rho)} - \frac{1}{6}\frac{\gamma^2-\gamma}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^2 - \frac{9}{32}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^3 -$$

$$\frac{1}{3}\frac{1}{\rho(1-\rho)+\gamma^2-\gamma} + \frac{2}{3}\frac{1}{\rho(1-\rho)+\frac{2}{9}(\gamma^2-\gamma)} < \frac{1}{\rho(1-\rho)} - \frac{1}{6}\frac{\gamma^2-\gamma}{\rho(1-\rho)} + \frac{3}{8}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^2 - \frac{11}{24}\left\{\frac{\gamma^2-\gamma}{\rho(1-\rho)}\right\}^3 \cdots$$

Since the first value of $\rho(1-\rho)$ is about $\frac{1}{100}$ we see that the geometric mean is a much closer approximation than either.

$$\sum \frac{1}{\rho(1-\rho)+\frac{1}{6}(\gamma^2-\gamma)} = \gamma\left\{\frac{1+\sqrt{(1-\gamma+\gamma^2)}}{2}\right\} \text{ and } \sum \frac{1}{\rho(1-\rho)+\frac{2}{9}(\gamma^2-\gamma)} = \gamma\left\{\frac{1+\sqrt{(1-3\gamma+3\gamma^2)}}{2}\right\}$$

Hence $\sum \frac{1}{\sqrt{\rho(1-\rho)(\rho-\gamma)(1-\rho-\gamma)}}$ lies between

(341) $\frac{1}{3}\chi(1) + \frac{2}{3}\chi\left\{\frac{1+\sqrt{(1-3\gamma+3\gamma^2)}}{2}\right\}$ and $\frac{1}{3}\chi(1) + \frac{1}{3}\chi\left\{\frac{1+\sqrt{(1-\gamma+\gamma^2)}}{2}\right\}$

and is also less than the geometric mean between these two.

66. In this and the following few sections it is always understood that p is the largest prime not greater than n. It can easily be shown that

(342) $$\int^{g(n)} \frac{dt}{t^s-1} - s\int \frac{x^{\frac{1}{2}}+x^{\frac{1}{3}}}{x^{1+s}(x^s-1)^2}dx = \frac{g(n)}{1-s} + \frac{g(n)^{1-s}}{1-2s} + \frac{x^{1-2s}}{1-3s} + \frac{x^{1-3s}}{1-4s} + \cdots + \frac{x^{1-ns}}{1-(n+1)s}$$
$$- \frac{2s\,x^{\frac{1}{2}-s}}{1-2s} - \frac{3s\,x^{\frac{1}{3}-s}}{1-3s} - \frac{4s\,x^{\frac{1}{4}-s}}{1-4s} + O(x^{\frac{1}{n}-s})$$

where $n = \left[2 + \frac{1}{2s}\right]$.

It follows from (230) and (342) that, if $s > 0$, then

(343) $$\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1}$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{g(1)\}^{1-s}}{1-s} + \frac{\{g(1)\}^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \frac{x^{1-4s}}{1-4s} + \cdots + \frac{x^{1-ns}}{1-ns}$$
$$- \frac{2s\,x^{\frac{1}{2}-s}}{1-2s} - \frac{3s\,x^{\frac{1}{3}-s}}{1-3s} - \frac{4s\,x^{\frac{1}{4}-s}}{1-4s} + S_s(x) + O\left(x^{\frac{1}{n}-s} + x^{\frac{1}{n}-s}\right)$$

where $n = \left[2 + \frac{1}{2s}\right]$

When $s = 1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{4}$ we must take the limit of the right hand side when s approaches $1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{4}$. We shall consider the following cases:

Case I $0 < s < \frac{1}{4}$

(344) $$\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1}$$
$$= \frac{\{g(1)\}^{1-s}}{1-s} + \frac{\{g(1)\}^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \frac{x^{1-4s}}{1-4s} + \cdots + \frac{x^{1-ns}}{1-ns} - \frac{2s\,x^{\frac{1}{2}-s}}{1-2s} - \frac{3s\,x^{\frac{1}{3}-s}}{1-3s}$$
$$+ S_s(x) + O(x^{\frac{1}{n}-s}), \text{ where } n = \left[2 + \frac{1}{2s}\right]$$

Case II. $s = \frac{1}{4}$.

(345) $$\frac{\log 2}{2^{\frac{1}{4}}-1} + \frac{\log 3}{3^{\frac{1}{4}}-1} + \frac{\log 5}{5^{\frac{1}{4}}-1} + \cdots + \frac{\log p}{p^{\frac{1}{4}}-1}$$
$$= \tfrac{4}{3}\{g(1)\}^{\frac{3}{4}} + 2\sqrt{\{g(1)\}} + 3x^{\frac{1}{4}} - 3x^{\frac{1}{6}} + \tfrac{1}{2}\log x + S_{\frac{1}{4}}(x) + O(1).$$

Case III. $s > \frac{1}{4}$

(346) $$\frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} \qquad (84)$$
$$= -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{g(1)\}^{1-s}}{1-s} + \frac{x^{1-2s} - 2s\,x^{\frac{1}{2}-s}}{1-2s} + \frac{x^{1-3s} - 3s\,x^{\frac{1}{3}-s}}{1-3s} + S_s(x) + O\left(x^{\frac{1}{n}-s}\right)$$

67. Making $s \to 1$ in 346, and remembering that

$$\lim_{s\to 1}\left\{\frac{\nu^{1-s}}{1-s} - \frac{\zeta'(s)}{\zeta(s)}\right\} = \log\nu - \gamma$$

where γ is the Eulerian constant, we have

$$(347)\quad \frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \frac{\log 5}{5-1} + \cdots + \frac{\log p}{p-1} = \log\vartheta(x) - \gamma + 2x^{-\frac{1}{2}} + \tfrac{2}{3}x^{-\frac{3}{2}} + S_1(x) + O(x^{-\frac{5}{2}})$$

From (332) we know that

$$(348)\quad N \times |S_1(x)| \leq 2+\gamma-\log(4\pi) = .046.$$

approximately, for all positive values of x.

When $s>1$, (346) reduces to

$$(349)\quad \frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{2s\,x^{\frac{1}{2}-s}}{2s-1} + \frac{3s\,x^{\frac{1}{3}-s}}{3s-1} + S_2(x) + O(x^{\frac{1}{4}-s})$$

Writing $O(x^{\frac{1}{2}-s})$ for $S_2(x)$ in 349, we see that, if $s>0$, then

$$(350)\quad \frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{x^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \cdots + \frac{x^{1-ns}}{1-ns} - \frac{2s\,x^{\frac{1}{2}-s}}{1-2s} + O(x^{\frac{1}{3}-s})$$

when $n = \left[1+\frac{1}{2s}\right]$

Now the following three cases arise:

Case I $\quad 0<s<\frac{1}{2}$

$$(351)\quad \frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} = \frac{\{\vartheta(x)\}^{1-s}}{1-s} + \frac{x^{1-2s}}{1-2s} + \frac{x^{1-3s}}{1-3s} + \cdots + \frac{x^{1-ns}}{1-ns} + O\left(x^{\frac{1}{2}-s}\right) \quad \text{where } n=\left[1+\tfrac{1}{2s}\right] \tag{352}$$

Case II $\quad s=\frac{1}{2}$

$$(352)\quad \frac{\log 2}{\sqrt{2}-1} + \frac{\log 3}{\sqrt{3}-1} + \frac{\log 5}{\sqrt{5}-1} + \cdots + \frac{\log p}{\sqrt{p}-1} = 2\sqrt{\vartheta(x)} + \tfrac{1}{2}\log x + O(1)$$

Case III $\quad s>\frac{1}{2}$

$$(353)\quad \frac{\log 2}{2^s-1} + \frac{\log 3}{3^s-1} + \frac{\log 5}{5^s-1} + \cdots + \frac{\log p}{p^s-1} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{\{\vartheta(x)\}^{1-s}}{1-s} + O(x^{\frac{1}{2}-s})$$

68. We shall now consider the product

$$(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})$$

It can easily be shown that

$$(354)\quad \int \frac{x^{a+bs}}{a+bs}\,ds = \frac{1}{b}\mathrm{Li}(x^{a+bs})$$

where $\mathrm{Li}(x)$ is the principal value of $\int_0^x \frac{dt}{\log t}$;

and that

$$(355)\quad \int S_0(x)\,ds = -\frac{S_1(x)}{\log x} + O\left\{\frac{x^{\frac{1}{2}-s}}{(\log x)^2}\right\}.$$

Now remembering (354) and (353) and integrating (349) with respect to s, we see that if $s>0$, then

$$(356)\quad \log\{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})\}$$

$$= -\log|\zeta(s)| - Li\{g(s)\}^{1-s} - \tfrac{1}{2}Li(x^{1-2s}) - \tfrac{1}{3}Li(x^{1-3s}) - \cdots$$
$$- \tfrac{1}{n}Li(x^{1-ns}) + \tfrac{1}{2}Li(x^{\frac{1}{2}-s}) - \frac{x^{\frac{1}{2}-s}}{\log x} + S_1(s) + O\left\{\frac{x^{\frac{1}{2}-\sigma}}{(\log x)^2}\right\}.$$

where $n = [1+\frac{1}{2s}]$. Now the following three cases arise.

Case I. $0 < s < \frac{1}{2}$

(357) $$\log\{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})\} \qquad (56)$$
$$= -Li\{g(s)\}^{1-s} - \tfrac{1}{2}Li(x^{1-2s}) - \tfrac{1}{3}Li(x^{1-3s}) - \cdots$$
$$- \tfrac{1}{n}Li(x^{1-ns}) + \frac{2s\,x^{\frac{1}{2}-s}}{(1-2s)\log x} - \frac{S_2(s)}{\log x} + O\left\{\frac{x^{\frac{1}{2}-\sigma}}{(\log x)^2}\right\}$$

where $n = [1+\frac{1}{2s}]$. Making $s \to \frac{1}{2}$ in (356) and remarking that

(358) $$\lim_{h\to 0}\{Li(1+h) - \log|h|\} = \gamma$$

where γ is the Eulerian constant, we have

Case II. $s = \frac{1}{2}$

(359) $$\frac{1}{(1-\frac{1}{\sqrt2})(1-\frac{1}{\sqrt3})(1-\frac{1}{\sqrt5})\cdots(1-\frac{1}{\sqrt p})} = -\sqrt2\,\zeta(\tfrac12)\exp\left\{Li\sqrt{g(s)} + \frac{1+\sqrt2}{\log x}S_1(s) + O\left(\frac{1}{(\log x)^2}\right)\right\}$$

It may be observed that

(360) $$-(\sqrt2-1)\zeta(\tfrac12) = \frac{1}{\sqrt1} - \frac{1}{\sqrt2} + \frac{1}{\sqrt3} - \frac{1}{\sqrt4} + \cdots$$

Case III. $s > \frac{1}{2}$.

(361) $$\frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})\cdots(1-p^{-s})} = |\zeta(s)|\exp\left[Li\{g(s)\}^{1-s} + \frac{2s\,x^{\frac{1}{2}-s}}{(2s-1)\log x} + \frac{S_2(s)}{\log x} + O\left\{\frac{x^{\frac{1}{2}-\sigma}}{(\log x)^2}\right\}\right]$$

Remarking (358) and making $s \to 1$ in (361) we obtain

(362) $$\frac{1}{(1-\frac12)(1-\frac13)(1-\frac15)\cdots(1-\frac1p)} = e^{\gamma}\left\{\log g(s) + \frac{2}{\sqrt x} + S_1(s) + O\frac{1}{\sqrt x\log x}\right\}$$

It follows from this and (347) that

(362) $$\frac{e^{-\gamma}}{(\frac12)(1-\frac13)(1-\frac15)\cdots(1-\frac1p)} = \gamma + \frac{\log 2}{2-1} + \frac{\log 3}{3-1} + \cdots + \frac{\log p}{p-1} + O\left(\frac{1}{\sqrt p\log p}\right).$$

Add MS. 9L.19 (23) (47) (87)

69. We shall consider the order of x_n. Putting $n=1$ in (319) we have

$$\varepsilon = \frac{\log(1+x_1^{-s})}{\log x_1} - 3,$$

and so

$$x_n^{\frac{\log(1+x_1^{-s})}{\log x_1}} = \frac{1-x_n^{-s(n+1)}}{1-x_n^{-sn}}. \tag{364}$$

Let

$$x_n = x_1^{t_n/n}.$$

Then we have

$$(1+x_1^{-s})^{t_n/n} = \frac{1-x_1^{-s t_n(1+\frac{1}{n})}}{1-x_1^{-s t_n}}.$$

From this we can easily deduce that

$$t_n = 1 + \frac{\log n}{s\log x_1} + O\left\{\frac{1}{(\log x_1)^2}\right\}.$$

Hence

$$x_n = x_1^{1/n}\left\{n^{1/(ns)} + O\left(\frac{1}{\log x_1}\right)\right\}; \tag{365}$$

and so

$$x_n \sim \left(n^{1/s}x_1\right)^{1/n}. \tag{366}$$

Putting $\lambda = 2$ in (311) we see that the greatest possible value of n is

$$a_2 = \frac{\log\frac{1}{\varepsilon}}{s\log 2} + O(1) = \frac{\log x_1}{\log 2} + \frac{\log\log x_1}{s\log 2} + O(1). \tag{367}$$

Again

$$\log N = \vartheta(x_1) + \vartheta(x_2) + \vartheta(x_3) + \cdots = \vartheta(x_1) + x_2 + O(x_1^{1/3}) \tag{368}$$

in virtue of (366). It follows from § 68 and the definition of $\Pi_n(x)$, that, if sn and $s(n+1)$ are not equal to 1, then

$$\Pi_n(x) = \left|\frac{\zeta(sn)}{\zeta\{s(n+1)\}}\right| e^{O(x^{1-sn})},$$

and consequently

$$\Pi_n(x_n) = \left|\frac{\zeta(sn)}{\zeta\{s(n+1)\}}\right| e^{O(x_1^{\frac{1}{n}-s})} \tag{369}$$

in virtue of (366). But if sn or $s(n+1)$ is unity, it can easily be shown that

hold [illegible] (88).

$$(3\cdot70)\qquad \Pi_{n-1}(x_{n-1})\,\Pi_n(x_n)\,\Pi_{n+1}(x_{n+1}) = \left|\frac{\zeta\{s(n-1)\}}{\zeta\{s(n+2)\}}\right| e^{O(x_1^{\frac{1}{2}-s})}.$$

70. We shall now consider the order of $\Sigma_{-s}(N)$ i.e. the maximum order of $\sigma_{-s}(N)$. It follows from (317) that, if $3s \neq 1$, then

$$(3\cdot71)\qquad \Sigma_{-s}(N) = \Pi_1(x_1)\,\Pi_2(x_2)\,|\zeta(3s)|\, e^{O(x_1^{\frac{1}{2}-s})},$$

in virtue of (367), (369) and (370). But if $3s = 1$, we can easily show, by using (362), that

$$(3\cdot72)\qquad \Sigma_{-s}(N) = \Pi_1(x_1)\,\Pi_2(x_2)\, e^{O(\log\log x_1)}.$$

It follows from §68 that

$$(3\cdot73)\qquad \log\Pi_1(x_1) = \log\left|\frac{\zeta(s)}{\zeta(3s)}\right| + \mathrm{Li}\{\vartheta(x_1)\}^{1-s} - \tfrac{1}{2}\mathrm{Li}\{\vartheta(x_1)\}^{1-2s} + \tfrac{1}{3}\mathrm{Li}\{\vartheta(x_1)\}^{1-3s} - \cdots\cdots - \frac{(-1)^n}{n}\mathrm{Li}\{\vartheta(x_1)\}^{1-ns} - \tfrac{1}{2}\mathrm{Li}(x_1^{\frac{1}{2}-s}) + \frac{x_1^{\frac{1}{2}-s} + S_3(x_1)}{\log x_1} + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\},$$

where $n = [1 + \frac{1}{2s}]$; and also that, if $3s \neq 1$, then

$$(3\cdot74)\qquad \log\Pi_2(x_2) = \log\left|\frac{\zeta(2s)}{\zeta(3s)}\right| + \mathrm{Li}(x_2^{1-2s}) + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\};$$

and when $3s = 1$

$$(3\cdot75)\qquad \log\Pi_2(x_2) = \mathrm{Li}(x_2^{1-2s}) + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}.$$

It follows from (371) — (375) that

$$(3\cdot76)\qquad \log\Sigma_{-s}(N) = \log|\zeta(s)| + \mathrm{Li}\{\vartheta(x_1)\}^{1-s} - \tfrac{1}{2}\mathrm{Li}\{\vartheta(x_1)\}^{1-2s} + \tfrac{1}{3}\mathrm{Li}\{\vartheta(x_1)\}^{1-3s} - \cdots\cdots - \frac{(-1)^n}{n}\mathrm{Li}\{\vartheta(x_1)\}^{1-ns} - \tfrac{1}{2}\mathrm{Li}(x_1^{\frac{1}{2}-s}) + \mathrm{Li}(x_2^{1-2s}) + \frac{x_1^{\frac{1}{2}-s} + S_3(x_1)}{\log x_1} + O\left\{\frac{x_1^{\frac{1}{2}-s}}{(\log x_1)^2}\right\}$$

where $n = [1 + \frac{1}{2s}]$. But from (368) it is clear that, if $m > 0$, then

$$\mathrm{Li}\{\vartheta(x_1)\}^{1-ms} = \mathrm{Li}\{\log N - x_2 + O(x_1^{1/3})\}^{1-ms}$$
$$= \mathrm{Li}\left\{(\log N)^{1-ms} - (1-ms)\,x_2(\log N)^{-ms} + O(x_1^{\frac{1}{3}-ms})\right\}$$
$$= \mathrm{Li}(\log N)^{1-ms} - \frac{x_2(\log N)^{-ms}}{\log\log N} + O(x_1^{\frac{1}{3}-ms})$$

(87)

By arguments similar to those of §4 we can show that

$$S_s(x_1) = S_s\left\{\log N + O\sqrt{x_1}(\log x_1)^2\right\} = S_s(\log N) + O\left\{x_1^{-s}(\log x_1)^2\right\}.$$

Hence

$$\log \Sigma_{-s}(N) = \log|\zeta(s)| + Li(\log N)^{1-s} - \tfrac{1}{2}Li(\log N)^{1-2s} + \tfrac{1}{3}Li(\log N)^{1-3s} - \cdots\cdots - \frac{(-1)^n}{n}Li(\log N)^{1-ns} - \tfrac{1}{2}Li(\log N)^{\frac{1}{2}-s} + \frac{(\log N)^{\frac{1}{2}-s} + S_s(\log N)}{\log\log N} + Li(x_2^{1-2s}) - \frac{x_2(\log N)^{-s}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\} \tag{377}$$

where $n = [1 + \frac{1}{2s}]$ and

$$x_2 = 2^{1/(2s)}\sqrt{x_1} + O\left(\frac{\sqrt{x_1}}{\log x_1}\right) = 2^{1/(2s)}\sqrt{\log N} + O\left\{\frac{\sqrt{\log N}}{\log\log N}\right\} \tag{378}$$

in virtue of (365).

71. Let us consider the order of $\Sigma_{-s}(N)$ in the following 3 cases.

Case I $\qquad 0 < s < \frac{1}{2}$.

Here we have

$$Li(\log N)^{\frac{1}{2}-s} = \frac{(\log N)^{\frac{1}{2}-s}}{(\frac{1}{2}-s)\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}.$$

$$Li(x_2^{1-2s}) = \frac{x_2^{1-2s}}{(1-2s)\log x_2} + O\left\{\frac{x_2^{1-2s}}{(\log x_2)^2}\right\} = \frac{2^{1/(2s)}(\log N)^{\frac{1}{2}-s}}{(1-2s)\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}.$$

$$\frac{x_2(\log N)^{-s}}{\log\log N} = \frac{2^{1/(2s)}(\log N)^{\frac{1}{2}-s}}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\}$$

It follows from these and (377) that

$$\log \Sigma_{-s}(N) = Li(\log N)^{1-s} - \tfrac{1}{2}Li(\log N)^{1-2s} + \tfrac{1}{3}Li(\log N)^{1-3s} - \cdots\cdots - \frac{(-1)^n}{n}Li(\log N)^{1-ns} - \frac{2s(2^{1/(2s)}-1)(\log N)^{\frac{1}{2}-s}}{(1-2s)\log\log N} + \frac{S_s(\log N)}{\log\log N} + O\left\{\frac{(\log N)^{\frac{1}{2}-s}}{(\log\log N)^2}\right\} \tag{379}$$

where $n = [1 + \frac{1}{2s}]$. Remembering (358) and (378) and making $s \to \frac{1}{2}$ in (377) we have

Case II $\quad S = \frac{1}{2}$.

$$\sum_{-\frac{1}{2}}(N) = \sqrt{2}\,\zeta(\tfrac{1}{2})\exp\left\{Li\sqrt{(\log N)} + \frac{2\log 2 - 1}{\log\log N} + \frac{S_{\frac{1}{2}}(\log N)}{\log\log N} - + O\left(\frac{1}{(\log\log N)^2}\right)\right\} \tag{38'}$$

Case III $\quad S > \frac{1}{2}$.

$$\sum_{-S}(N) = \left|\zeta(S)\right|\exp\left\{Li\,(\log N)^{1-S} + \frac{2S(2^{1/(2S)}-1)}{2S-1}\frac{(\log N)^{\frac{1}{2}-S}}{\log\log N} + \cdots + \frac{S_S(\log N)}{\log\log N} - + O\left\{\frac{(\log N)^{\frac{1}{2}-S}}{(\log\log N)^2}\right\}\right) \tag{38''}$$

Now making $S > 1$ in this we have

$$\sum_{-1}(N) = e^{\gamma}\left\{\log\log N + \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} - + S_1(\log N) + O\left(\sqrt{(\log N)}^{-1}\log\log N\right)\right\} \tag{38}$$

Hence

$$\overline{\lim}\left\{\sum_{-1}(N) - e^{\gamma}\log\log N\right\}\sqrt{(\log N)} \leq e^{\gamma}(2\sqrt{2} + \gamma - \log 4\pi)$$

$$= 1.558$$

approximately and

$$\underline{\lim}\left\{\sum_{-1}(N) - e^{\gamma}\log\log N\right\}\sqrt{(\log N)} \geq e^{\gamma}(2\sqrt{2} - 4 - \gamma + \log 4\pi)$$

$$= -1.393$$

approximately.

The maximum order of $\sigma_{-S}(N)$ is easily obtained by multiplying the values of $\sum_{-S}(N)$ by N^S. It may be interesting to see that $x_n \to x_1^{1/n}$ as $S \to \infty$; and ultimately N assumes the form

$$e^{\Theta(x_1)} + \Theta(\sqrt{x_1}) + \Theta(x_1^{1/3}) + \cdots,$$

That is to say the form of a generalised superior highly Composite number approaches that of the least Common multiple of the natural numbers when S becomes infinitely large.

The maximum order of $\sigma_{-S}(N)$ without assuming the prime number theorem is obtained by changing N to $Ne^{O(1)}$ in all the preceding results. In particular

$$\sum_{-1}(N) = e^{\gamma}\left\{\log\log N + O(1)\right\}. \tag{38'''}$$

log

(71)

72. Let

$$(1 + 2\psi + 2\psi^4 + 2\psi^9 + \cdots)^4$$
$$= 1 + 8\{Q_4(1)\psi + Q_4(2)\psi^2 + Q_4(3)\psi^3 + \cdots\}.$$

Then, by means of elliptic functions, we can show that

(384) $$Q_4(1)\psi + Q_4(2)\psi^2 + Q_4(3)\psi^3 + \cdots$$
$$= \frac{\psi}{1-\psi} + \frac{3\psi^2}{1+\psi^2} + \frac{3\psi^3}{1-\psi^3} + \frac{4\psi^4}{1+\psi^4} + \cdots$$
$$= \frac{\psi}{1-\psi} + \frac{2\psi^2}{1-\psi^2} + \frac{3\psi^3}{1-\psi^3} + \frac{4\psi^4}{1-\psi^4} + \cdots$$
$$- \left(\frac{4\psi^4}{1-\psi^4} + \frac{8\psi^8}{1-\psi^8} + \frac{12\psi^{12}}{1-\psi^{12}} + \cdots\right)$$

But

$$\frac{\psi}{1-\psi} + \frac{2\psi^2}{1-\psi^2} + \frac{3\psi^3}{1-\psi^3} + \cdots$$
$$= \sigma_1(1)\psi + \sigma_1(2)\psi^2 + \sigma_1(3)\psi^3 + \cdots$$

It follows that

(385) $$Q_4(N) \leqq \sigma_1(N)$$

for all values of N. It also follows from (384) that

(386) $$(1-4^{1-s})\zeta(s)\zeta(s-1) = 1^{-s}Q_4(1) + 2^{-s}Q_4(2) + 3^{-s}Q_4(3) + \cdots$$

Let $$N = 2^{a_2}3^{a_3}5^{a_5}\cdots p^{a_p}$$

where $a_\lambda \geqq 0$. Then, ~~if N is not a multiple of 4, then~~ the coefficient of ψ^N in

$$\frac{\psi}{1-\psi} + \frac{2\psi^2}{1-\psi^2} + \frac{3\psi^3}{1-\psi^3} + \cdots$$

is $$N\cdot\frac{1-2^{-a_2-1}}{1-2^{-1}}\cdot\frac{1-3^{-a_3-1}}{1-3^{-1}}\cdot\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}};$$

and that in

$$\frac{4\psi^4}{1-\psi^4} + \frac{8\psi^8}{1-\psi^8} + \frac{12\psi^{12}}{1-\psi^{12}} + \cdots$$

is 0 when N is not a multiple of 4 and

$$N\cdot\frac{1-2^{-a_2+1}}{1-2^{-1}}\cdot\frac{1-3^{-a_3-1}}{1-3^{-1}}\cdot\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}$$

when N is a multiple of 4. From this and (384) it follows that, if

(92)

N is not a multiple of 4, then

$$(387)\qquad Q_4(N) = N\cdot\frac{1-2^{-a_2-1}}{1-2^{-1}}\cdot\frac{1-3^{-a_3-1}}{1-3^{-1}}\cdot\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}};$$

and if N is a multiple of 4, then

$$(388)\qquad Q_4(N) = 3N\cdot\frac{2^{-a_2-1}}{1-2^{-1}}\cdot\frac{1-3^{-a_3-1}}{1-3^{-1}}\cdot\frac{1-5^{-a_5-1}}{1-5^{-1}}\cdots\frac{1-p^{-a_p-1}}{1-p^{-1}}.$$

It is easy to see from (387) and (388) that, in order that $Q_4(N)$ should be of maximum order, a_2 must be 1. From (387) we see that the maximum order of $Q_4(N)$ is

$$(389)\qquad \frac{3}{4}e^{\gamma}\left\{\log\log N + \frac{2(\sqrt{2}-1)}{\sqrt{(\log N)}} + S_1(\log N) + O\left(\frac{1}{\sqrt{(\log N)}\log\log N}\right)\right\}$$

$$= \frac{3}{4}e^{\gamma}\left\{\log\log N + O\left(\frac{1}{\sqrt{(\log N)}}\right)\right\}.$$

It may be observed that, if N is not a multiple of 4, then

$$Q_4(N) = \sigma_1(N);$$

and if N is a multiple of 4, then

$$Q_4(N) = \frac{3\sigma_1(N)}{2^{a_2+1}-1}.$$

73. Let

$$(1+2v+2v^4+2v^9+\cdots)^6$$

$$= 1+12\left\{Q_6(1)v + Q_6(2)v^2 + Q_6(3)v^3 + \cdots\right\}.$$

Then, by means of Elliptic functions, we can show that

$$(390)\qquad Q_6(1)v + Q_6(2)v^2 + Q_6(3)v^3 + \cdots$$

$$= \frac{4}{3}\left(\frac{1^2v}{1+v^2} + \frac{2^2v^2}{1+v^4} + \frac{3^2v^3}{1+v^6} + \cdots\right)$$

$$- \frac{1}{3}\left(\frac{1^2v}{1-v} - \frac{3^2v^3}{1-v^3} + \frac{5^2v^5}{1-v^5} - \cdots\right).$$

But

$$\frac{5}{3}\left\{\sigma_2(1)v + \sigma_2(2)v^2 + \sigma_2(3)v^3 + \cdots\right\}$$

$$= \frac{4}{3}\left\{\frac{1^2v}{1-v} + \frac{2^2v^2}{1-v^2} + \frac{3^2v^3}{1-v^3} + \cdots\right\}$$

$$+ \frac{1}{3}\left\{\frac{1^2v}{1-v} + \frac{2^2v^2}{1-v^2} + \frac{3^2v^3}{1-v^3} + \cdots\right\}.$$

It follows that

$$(391)\qquad Q_6(N) \leqq \frac{5\sigma_2(N)-2}{3},$$

for all values of N. It also follows from (390) that

$$(392)\qquad \frac{4}{3}\zeta(s-2)\zeta_1(s) - \frac{1}{3}\zeta(s)\zeta_1(s-2) = 1^{-s}Q_6(1) + 2^{-s}Q_6(2) + 3^{-s}Q_6(3) + \cdots$$

Let $N = 2^{a_2} 3^{a_3} 5^{a_5} \cdots p^{a_p}$, (93)

where $a_p \neq 0$. Then from (390) we can show, as in the previous section, that if $2^{-a_2}N$ be of the form $4n+1$, then

$$(375)\quad Q_6(N) = N^2 \frac{1-(2^2)^{-a_2-1}}{1-2^{-2}} \cdot \frac{1-(-3^2)^{-a_3-1}}{1+3^{-2}} \cdot \frac{1-(5^2)^{-a_5-1}}{1-5^{-2}} \cdot \cdots \cdot \frac{1-\{(-1)^{\frac{p-1}{2}} p^2\}^{-a_p-1}}{1-(-1)^{\frac{p-1}{2}} p^{-2}};$$

and if $2^{-a_2}N$ be of the form $4n-1$, then

$$(376)\quad Q_6(N) = N^2 \frac{1+(2^2)^{-a_2-1}}{1-2^{-2}} \cdot \frac{1-(-3^2)^{-a_3-1}}{1+3^{-2}} \cdot \frac{1-(5^2)^{-a_5-1}}{1-5^{-2}} \cdot \cdots \cdot \frac{1-\{(-1)^{\frac{p-1}{2}} p^2\}^{-a_p-1}}{1-(-1)^{\frac{p-1}{2}} p^{-2}};$$

It follows from (373) and (374) that, in order that $Q_6(N)$ should be of maximum order, $2^{-a_2}N$ must be of the form $4n-1$ and $a_1, a_3, a_7, a_{11}, \ldots$ must be 0; $3, 7, 11, \ldots$ being primes of the form $4n-1$. But all these cannot be satisfied at the same time since $2^{-a_2}N$ cannot be of the form $4n-1$, when $a_3, a_7, a_{11}, \ldots$ are all zeros. So let us retain a single prime of the form $4n-1$ in the end, that is to say, the largest prime of the form $4n-1$ not exceeding p. Thus we see that, in order that $Q_6(N)$ should be of maximum order, N must be of the form

$$5^{a_5} \cdot 13^{a_{13}} \cdot 17^{a_{17}} \cdots \cdots p^{a_p} \cdot p',$$

where p is a prime of the form $4n+1$ and p' is the prime of the form $4n-1$ next above or below p; and consequently

$$Q_6(N) = \frac{5}{3} N^2 \cdot \frac{1-5^{-2(a_5+1)}}{1-5^{-2}} \cdot \frac{1-13^{-2(a_{13}+1)}}{1-13^{-2}} \cdots \cdots \frac{1-p^{-2(a_p+1)}}{1-p^{-2}} \left\{1-(p')^{-2}\right\}.$$

From this we can show that the maximum order of $Q_6(N)$ is

$$(377)\quad \frac{5N^2 e^{\frac{1}{2}Li\left(\frac{1}{2\log N}\right)} + O\left\{\frac{1}{\log N \log\log N \sqrt{(\log N)}}\right\}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots\cdots}$$

$$= \frac{5N^2\left\{1+\frac{1}{2}Li\left(\frac{1}{2\log N}\right) + O\frac{1}{\log N \log\log N \sqrt{(\log N)}}\right\}}{3\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{13^2}\right)\left(1-\frac{1}{17^2}\right)\left(1-\frac{1}{29^2}\right)\cdots\cdots}$$

where $5, 13, 17, \ldots$ are the primes of the form $4n+1$.

74. Let $(1+2q+2q^4+2q^9+\cdots)^8$

$$= 1 + 16\{\varphi_8(1)q + \varphi_8(2)q^2 + \varphi_8(3)q^3 + \cdots\}$$

Then, by means of Elliptic functions, we can show that

$$\varphi_8(1)q + \varphi_8(2)q^2 + \varphi_8(3)q^3 + \cdots \tag{370}$$
$$= \frac{1^3 q}{1+q} + \frac{2^3 q^2}{1+q^2} + \frac{3^3 q^3}{1+q^3} + \frac{4^3 q^4}{1+q^4} + \cdots$$

But $\sigma_3(1)q + \sigma_3(2)q^2 + \sigma_3(3)q^3 + \cdots$

$$= \frac{1^3 q}{1-q} + \frac{2^3 q^2}{1-q^2} + \frac{3^3 q^3}{1-q^3} + \cdots$$

It follows that

$$\varphi_8(N) \leqq \sigma_3(N) \tag{377}$$

for all values of N. It can also be shown from (396) that

$$(1 - 2^{1-s} + 4^{2-s})\zeta(s)\zeta(s-3) = \varphi_8(1)q + \varphi_8(2)q^2 + \varphi_8(3)q^3 + \cdots \tag{378}$$

Let $N = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdots p^{a_p}$,

where $a_2 \geqq 0$. Then from (396) we can easily show that, if N is odd, then

$$\varphi_8(N) = N^3 \cdot \frac{1 - 2^{-3(a_2+1)}}{1 - 2^{-3}} \cdot \frac{1 - 3^{-3(a_3+1)}}{1 - 3^{-3}} \cdots \frac{1 - p^{-3(a_p+1)}}{1 - p^{-3}} \tag{379}$$

and if N is even then

$$\varphi_8(N) = N^3 \cdot \frac{1 - 15 \cdot 2^{-3(a_2+1)}}{1 - 2^{-3}} \cdot \frac{1 - 3^{-3(a_3+1)}}{1 - 3^{-3}} \cdots \frac{1 - p^{-3(a_p+1)}}{1 - p^{-3}} \tag{380}$$

Hence the maximum order of $\varphi_8(N)$ is

$$\zeta(3) N^3 e^{Li(\log N)^{-2}} + O\left(\frac{(\log N)^{-5/2}}{\log\log N}\right)$$
$$= \zeta(3) N^3 \left\{1 + Li(\log N)^{-2} + O\left(\frac{(\log N)^{-5/2}}{\log\log N}\right)\right\},$$

or more precisely

$$\zeta(3) N^3 \left\{1 + Li(\log N)^{-2} + \frac{6(2^{1/2}-1)}{5}(\log N)^{-5/2} + \frac{S_3(\log N)}{\log\log N} + O\left(\frac{(\log N)^{-5/2}}{(\log\log N)^2}\right)\right\} \tag{401}$$

(95)

75. There are of course results corresponding to those of §§72-74 for the various powers of $\overline{Q}$, where

$$\overline{Q} = 1 + 6\left(\frac{v}{1-v} - \frac{v^2}{1-v^2} + \frac{v^4}{1-v^4} - \frac{v^5}{1-v^5} + \cdots\right).$$

Thus for example

(402) $$(\overline{Q})^2 = 1 + 12\left(\frac{v}{1-v} + \frac{2v^2}{1-v^2} + \frac{4v^4}{1-v^4} + \frac{5v^5}{1-v^5} + \cdots\right)$$

(403) $$(\overline{Q})^3 = 1 - 9\left(\frac{1^2v}{1-v} - \frac{2^2v^2}{1-v^2} + \frac{4^2v^4}{1-v^4} - \frac{5^2v^5}{1-v^5} + \cdots\right)$$
$$+ 27\left(\frac{1^2v}{1+v+v^2} + \frac{2^2v^2}{1+v^2+v^4} + \frac{3^2v^3}{1+v^3+v^6} + \cdots\right)$$

(404) $$(\overline{Q})^4 = 1 + 24\left(\frac{1^3v}{1-v} + \frac{2^3v^2}{1-v^2} + \frac{3^3v^3}{1-v^3} + \cdots\right)$$
$$+ 8\left(\frac{3^3v^3}{1-v^3} + \frac{6^3v^6}{1-v^6} + \frac{9^3v^9}{1-v^9} + \cdots\right)$$

The number of ways in which a number can be expressed in the forms m^2+2n^2, $k^2+l^2+2m^2+2n^2$, m^2+3n^2, and $k^2+l^2+3m^2+3n^2$ can be found from the following formulae.

(405) $$(1+2v+2v^4+2v^9+\cdots\cdots)(1+2v^2+2v^8+2v^{18}+\cdots)$$
$$= 1+2\left(\frac{v}{1-v} + \frac{v^3}{1-v^3} - \frac{v^5}{1-v^5} - \frac{v^7}{1-v^7} + \cdots\right).$$

(406) $$(1+2v+2v^4+2v^9+\cdots)^2(1+2v^2+2v^8+2v^{18}+\cdots)^2$$
$$= 1+4\left(\frac{v}{1-v^2} + \frac{2v^2}{1-v^4} + \frac{3v^3}{1-v^6} + \frac{4v^4}{1-v^8} + \cdots\right),$$

(407) $$(1+2v+2v^4+2v^9+\cdots)(1+2v^3+2v^{12}+2v^{27}+\cdots)$$
$$= 1+2\left(\frac{v}{1-v} - \frac{v^2}{1+v^2} + \frac{v^4}{1+v^4} - \frac{v^5}{1-v^5} + \frac{v^7}{1-v^7} - \cdots\right).$$

(408) $$(1+2v+2v^4+2v^9+\cdots)^2(1+2v^3+2v^{12}+2v^{27}+\cdots)^2$$
$$= 1+4\left(\frac{v}{1+v} + \frac{2v^2}{1-v^3} + \frac{4v^4}{1-v^4} + \frac{5v^5}{1+v^5} + \frac{7v^7}{1+v^7} + \cdots\right),$$

where 1, 2, 4, 5, ... are the natural numbers without the multiple of 3.

N is highly composite

$$N = e^{\vartheta(p_1) + \vartheta(p_2) + \vartheta(p_3) + \cdots}$$

$$d(N) = 2^{\pi(p_1)} \left(\tfrac{3}{2}\right)^{\pi(p_2)} \left(\tfrac{5}{4}\right)^{\pi(p_3)} \cdots$$

Then $$N' = N e^{\vartheta(q_1) + \vartheta(q_2) - \vartheta(p_1) - \vartheta(p_2)}$$

$$d(N') \geq d(N) \cdot 2^{\pi(q_1) - \pi(p_1)} \left(1 + \tfrac{1}{\lambda}\right)^{\pi(q_2) - \pi(p_2)}$$

First let us suppose that $q_1 > p_1$, $q_2 < p_2$ and

$$\pi(q_1) = \pi(p_1) + n$$

$$\pi(q_2) = \pi(p_2) - \left[\frac{n \log 2}{\log(1 + \frac{1}{\lambda})}\right]$$

Then we have $d(N') \geq d(N)$ and so $N' \geq N$. That is to say

$$\vartheta(q_1) + \vartheta(q_2) \geq \vartheta(p_1) + \vartheta(p_2).$$

Now if $\alpha > \beta$, it is easy to see that

$$\{\pi(\alpha) - \pi(\beta)\} \log \alpha \geq \vartheta(\alpha) - \vartheta(\beta) \geq \{\pi(\alpha) - \pi(\beta)\} \log \beta$$

or $$\{\pi(q_1) - \pi(p_1)\} \log q_1 \geq \{\pi(p_2) - \pi(q_2)\} \log q_2,$$

or $$n \log q_1 \geq \left[\frac{n \log 2}{\log(1 + \frac{1}{\lambda})}\right] \log q_2$$

Now it is easy to show that, if $\pi(\alpha) = \pi(\beta) + \kappa$ and $\kappa = o\left(\frac{\beta}{\log \beta}\right)$, then

$$\log \alpha = \log \beta + O(1)$$

Hence $$\log q_1 = \log p_1 + O(1), \quad \log q_2 = \log p_2 + O(1),$$

and so $$n \{\log p_1 + O(1)\} \geq \left[\frac{n \log 2}{\log(1 + \frac{1}{\lambda})}\right] \{\log p_2 + O(1)\},$$

or $$\log p_1 + O(1) \geq \left\{\frac{\log 2}{\log(1 + \frac{1}{\lambda})} + O(\tfrac{1}{n})\right\} \{\log p_2 + O(1)\}.$$

In other words $$\frac{\log p_1}{\log 2} - \frac{\log p_2}{\log(1 + \frac{1}{\lambda})} \geq O\left(n + \frac{\log p_2}{\log 2}\right)$$

Now we can choose n so that $\log p_2 = O(n\lambda)$. Hence

Add this to 9.4.3(58)

$$\frac{\log p_1}{\log 2} - \frac{\log p_n}{\log(1+\frac{1}{n})} \geqslant O(1)$$

If we now suppose that $q_1 < p_1$, $q_n > p_n$, and

$$\pi(q_1) = \pi(p_1) - n - 1$$

$$\pi(q_n) = \pi(p_n) + \left[\frac{n \log 2}{\log(1+\frac{1}{n})}\right]$$

and proceeding as before we can show that

$$\frac{\log p_1}{\log 2} - \frac{\log p_n}{\log(1+\frac{1}{n})} \leq O(1)$$

It follows that

(1)
$$\frac{\log p_n}{\log(1+\frac{1}{n})} = \frac{\log p_1}{\log 2} + O(1).$$

If $N = e^{\vartheta(\alpha) + \vartheta(\beta) + \vartheta(\gamma) + \dots}$

then $d(N) \geqslant 2^{\pi(\alpha)} \left(\frac{3}{2}\right)^{\pi(\beta)} \left(\frac{5}{3}\right)^{\pi(\gamma)} \dots$

If $\alpha \geqslant \beta \geqslant \gamma \geqslant \dots$ then only

$$d(N) = 2^{\pi(\alpha)} \left(\frac{3}{2}\right)^{\pi(\beta)} \left(\frac{5}{3}\right)^{\pi(\gamma)} \dots$$

Let N be highly composite such that

$$N = e^{\vartheta(p_1) + \vartheta(p_2) + \vartheta(p_3) + \dots}$$

and $d(N) = 2^{\pi(p_1)} \left(\frac{3}{2}\right)^{\pi(p_2)} \left(\frac{5}{3}\right)^{\pi(p_3)} \dots$

Let $N' = N e^{\vartheta(q_n) + \vartheta(q_s) - \vartheta(p_r) - \vartheta(p_s)}$

Then $d(N') \geqslant d(N) \left(1 + \frac{1}{n}\right)^{\pi(q_n) - \pi(p_r)} \left(1 + \frac{1}{s}\right)^{\pi(q_s) - \pi(p_s)}$

Let us suppose that

$$q_n > p_r, \quad q_s < p_s,$$

and

$$(1) \qquad \begin{cases} \pi(q_n) = \pi(p_r) + n, \\ \pi(q_s) = \pi(p_s) - \left[\dfrac{n \log\left(1 + \frac{1}{n}\right)}{\log\left(1 + \frac{1}{s}\right)} \right]. \end{cases}$$

We have then

$$d(N') \geqslant d(N)$$

and so $N' \geqslant N$.

That is to say

$$\vartheta(q_n) + \vartheta(q_s) \geqslant \vartheta(p_r) + \vartheta(p_s).$$

But if $\alpha > \beta$, then

$$\{\pi(\alpha) - \pi(\beta)\} \log \alpha \geqslant \vartheta(\alpha) - \vartheta(\beta) \geqslant \{\pi(\alpha) - \pi(\beta)\} \log \beta.$$

Hence

$$\left\{\pi(p_n) - \pi(p_{n-1})\right\} \log p_n \geq \left\{\pi(p_s) - \pi(p_{s-1})\right\} \log p_s$$

or
$$n \log p_n \geq \left[\begin{matrix} n \log\left(1 + \frac{1}{n}\right) \\ + \log\left(1 + \frac{1}{n}\right) \end{matrix}\right] \log p_s .$$

Now we have to express p_n and p_s in terms of n and s respectively.

$$f(x, y) = 1 + x\left(y + \frac{1}{y}\right) + x^2\left(y^2 + \frac{1}{y^2}\right) + x^3\left(y^3 + \frac{1}{y^3}\right) + \cdots$$

$$\frac{f\left(\sqrt{ab}, n\sqrt{\frac{b}{a}}\right)}{f\left(-\sqrt{ab}, -\sqrt{\frac{b}{a}}\right) f\left(\sqrt{ab}, n\sqrt{ab}\right)} \left\{(1-ab)(1-a^2b^2)(1-a^3b^3)\cdots\right\}^3 .$$

$$= \frac{1}{1+n} + \left(\frac{a}{n+ab} + \frac{b}{1+nab}\right) + \left(\frac{a^2}{n+a^2b^2} + \frac{b^2}{1+na^2b^2}\right)$$

$$+ \left(\frac{a^3}{n+a^3b^3} + \frac{b^3}{1+na^3b^3}\right) + \cdots \qquad -$$

This is 10. p50 but for section 5
onwards is not published

(1)

1. Let

$$\phi(\alpha,\beta) = \left\{1+\left(\frac{\alpha+\beta}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{3+\alpha}\right)^3\right\}\cdots\cdots$$

It can easily be shown that

$$(1)\quad \left\{1+\left(\frac{\alpha+\beta}{m+\alpha}\right)^3\right\}\left\{1+\left(\frac{\alpha+\beta}{m+\beta}\right)^3\right\} = \frac{\left(1+\frac{\alpha+2\beta}{m}\right)\left(1+\frac{2\alpha+\beta}{m}\right)}{\left(1+\frac{\alpha}{m}\right)^3\left(1+\frac{\beta}{m}\right)^3}\times$$

$$\left\{1-\left(\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2m}\right)^2\right\}\left\{1-\left(\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2m}\right)^2\right\};$$

$$(2)\quad \prod_{m=1}^{m=\infty}\frac{\left(1+\frac{\alpha+2\beta}{m}\right)\left(1+\frac{2\alpha+\beta}{m}\right)}{\left(1+\frac{\alpha}{m}\right)^3\left(1+\frac{\beta}{m}\right)^3} = \frac{\left\{\Gamma(1+\alpha)\,\Gamma(1+\beta)\right\}^3}{\Gamma(1+\alpha+2\beta)\,\Gamma(1+\beta+2\alpha)};$$

$$(3)\quad \prod_{m=1}^{m=\infty}\left\{1-\left(\frac{(\alpha-\beta)+i(\alpha+\beta)\sqrt{3}}{2m}\right)^2\right\}\left\{1-\left(\frac{(\alpha-\beta)-i(\alpha+\beta)\sqrt{3}}{2m}\right)^2\right\}$$

$$= \frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}.$$

It follows from (1), (2) and (3) that

$$(4)\quad \phi(\alpha,\beta)\,\phi(\beta,\alpha) = \frac{\left\{\Gamma(1+\alpha)\,\Gamma(1+\beta)\right\}^3}{\Gamma(1+\alpha+2\beta)\,\Gamma(1+\beta+2\alpha)}\cdot\frac{\cosh\pi(\alpha+\beta)\sqrt{3}-\cos\pi(\alpha}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}$$

But $\phi(\alpha,\beta)/\phi(\beta,\alpha)$ can be expressed in finite terms if α be any integer. It follows from (4) that, if $\alpha-\beta$ be any integer, then $\phi(\alpha,\beta)$ can be expressed in finite terms. That is to say

$$(5)\quad \left\{1+\left(\frac{x}{m+1}\right)^3\right\}\left\{1+\left(\frac{x}{m+2}\right)^3\right\}\left\{1+\left(\frac{x}{m+3}\right)^3\right\}\cdots\cdots$$

can be expressed in finite terms, if $x-2m$ be any intego

(2)

2. Suppose now that $\alpha = \beta$ in (4). We obtain

$$(6)\quad \left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\cdots\cdots$$
$$= \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)}\cdot\frac{\sinh \pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}$$

Similarly supposing that $\beta = \alpha+1$ in (4) we obtain

$$(7)\quad \left\{1+\left(\frac{1+2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{1+2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{1+2\alpha}{3+\alpha}\right)^3\right\}\cdots\cdots$$
$$= \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\cdot\frac{\cosh \pi(\tfrac{1}{2}+\alpha)\sqrt{3}}{\pi}.$$

Since

$$\left\{1+\left(\frac{\alpha}{m}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2m+\alpha}\right)^2\right\} = \frac{1+\frac{\alpha}{m}}{\left(1+\frac{\alpha}{2m}\right)^2}\left(1+\frac{\alpha^3}{m^3}+\frac{\alpha^4}{m^4}\right)$$

it is easy to see that

$$(8)\quad \left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\left(1+\frac{\alpha^3}{4^3}\right)\cdots\cdots$$
$$\times\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{6+\alpha}\right)^2\right\}\cdots\cdots$$
$$= \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})}\cdot\frac{\cosh \pi\alpha\sqrt{3}-\cos\pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}.$$

3. It is known that, if the real part of α is positive, then

$$(9)\quad \log\Gamma(\alpha) = (\alpha-\tfrac{1}{2})\log\alpha - \alpha + \tfrac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\tan^{-1}\frac{x}{\alpha}}{e^{2\pi x}-1}\,dx.$$

From this we can easily show that, if the real part of α is positive, then

$$(10)\quad \tfrac{1}{2}\log(2\pi\alpha) + \log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)\cdots\cdots\right\}$$
$$= \log\left(\frac{\cosh\pi\alpha\sqrt{3}-\cos\pi\alpha}{\pi\alpha}\right) - \frac{\pi\alpha}{\sqrt{3}} + 2\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{\alpha}\right)^3}{e^{2\pi x}-1}\,dx$$

Hence we see that

$$\int_0^\infty \frac{\tan^{-1}x^3}{e^{2\pi\alpha x}-1}\,dx$$

Add [illegible] (3)

can be expressed in finite terms for all positive integral values of α. Thus for example

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{2\pi x}-1}\,dx = \tfrac{1}{4}\log 2\pi - \tfrac{1}{2}\log(1+e^{-\pi\sqrt{3}}) - \frac{\pi}{4\sqrt{3}};$$

$$\int_0^\infty \frac{\tan^{-1} x^3}{e^{4\pi x}-1}\,dx = \tfrac{1}{8}\log 12\pi - \tfrac{1}{2}\log(1-e^{-2\pi\sqrt{3}}) - \frac{\pi}{4\sqrt{3}};$$

and so on.

4. It is easy to see that

$$(11)\quad \frac{1^2}{1^3+\alpha^3} - \frac{2^2}{2^3+\alpha^3} + \frac{3^2}{3^3+\alpha^3} - \frac{4^2}{4^3+\alpha^3} + \cdots$$

$$= \tfrac{1}{3}\left(\frac{1}{1+\alpha} - \frac{1}{2+\alpha} + \frac{1}{3+\alpha} - \frac{1}{4+\alpha} + \cdots\right)$$

$$+ \frac{4}{3}\left\{\frac{2-\alpha}{(2-\alpha)^2+3\alpha^2} - \frac{4-\alpha}{(4-\alpha)^2+3\alpha^2} + \frac{6-\alpha}{(6-\alpha)^2+3\alpha^2} - \cdots\right\}.$$

Since

$$\frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \cdots = \frac{\pi}{4}\operatorname{Sech}\frac{\pi x}{2},$$

we see that the left hand side of (11) can be expressed in finite terms if α is any odd integer. For example

$$\frac{1^2}{1^3+1} - \frac{2^2}{2^3+1} + \frac{3^2}{3^3+1} - \cdots = \tfrac{1}{3}(1-\log 2 + \pi\operatorname{Sech}\tfrac{1}{2}\pi\sqrt{3}).$$

Again, if $\alpha > 0$, then

$$(12)\quad \int_0^\infty \frac{x^{6-\alpha}}{\sinh \pi x}\cdot\frac{dx}{\alpha^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2} + \sum_{n=1}^{n=\infty}\frac{(-1)^n}{n^2+x^2}\right\}\frac{x^6\,dx}{\alpha^6+x^6}$$

$$= \tfrac{1}{3}\left(\frac{1}{\alpha} - \frac{1}{1+\alpha} + \frac{1}{2+\alpha} - \frac{1}{3+\alpha} + \cdots\right)$$

$$- \frac{4}{3}\left\{\frac{2+\alpha}{(2+\alpha)^2+3\alpha^2} - \frac{4+\alpha}{(4+\alpha)^2+3\alpha^2} + \frac{6+\alpha}{(6+\alpha)^2+3\alpha^2} - \cdots\right\}.$$

Hence the left-hand side of (12) can be expressed in finite terms if α is any odd integer. For example

$$\int_0^\infty \frac{x^{5-\alpha}}{\sinh \pi x}\cdot\frac{dx}{1+x^6} = \frac{1}{3}\left(\log 2 - 1 + \pi\operatorname{Sech}\tfrac{1}{2}\pi\sqrt{3}\right).$$

(4)

5. If 2α be a positive integer, then it can easily be shown that

$$(13)\quad \frac{1}{1^4+4\alpha^4} + \frac{2}{2^4+4\alpha^4} + \frac{3}{3^4+4\alpha^4} + \cdots\cdots$$
$$= \frac{1}{4\alpha^2}\left\{ \frac{1}{(1-\alpha)^2+\alpha^2} + \frac{1}{(2-\alpha)^2+\alpha^2} + \frac{1}{(2-\alpha)^2+\alpha^2} + \cdots \text{ to } 2\alpha \text{ terms}\right\};$$

and

$$(14)\quad \int_0^\infty \frac{x}{e^{2\pi x}-1}\cdot\frac{dx}{4\alpha^4+x^4} = \frac{\pi}{8\alpha^2}\frac{(-1)^{2\alpha}}{e^{2\pi\alpha}+(-1)^{2\alpha+1}}$$
$$+ \frac{1}{8\alpha}\left\{ \frac{1}{(1-\alpha)^2+\alpha^2} + \frac{1}{(2-\alpha)^2+\alpha^2} + \frac{1}{(3-\alpha)^2+\alpha^2} + \cdots \text{to } 2\alpha \text{ terms}\right\}$$

It can easily be shown by the theory of residues, that

$$(15)\quad \frac{1}{16\pi\alpha^4} + \frac{\coth\pi}{1^4+4\alpha^4} + \frac{2\coth 2\pi}{2^4+4\alpha^4} + \frac{3\coth 3\pi}{3^4+4\alpha^4} + \cdots$$
$$= \frac{\pi}{8\alpha^2}\cdot\frac{\cosh 2\pi\alpha + \cos 2\pi\alpha}{\cosh 2\pi\alpha - \cos 2\pi\alpha}.$$

It follows from (13) and (15) that, if 2α be a positive integer, then

$$(16)\quad \frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^4+4\alpha^4} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^4+4\alpha^4} + \frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^4+4\alpha^4} + \cdots$$
$$= \frac{\pi}{16\alpha^2}\cdot\left\{\frac{e^{2\pi\alpha}+(-1)^{2\alpha}}{e^{2\pi\alpha}-(-1)^{2\alpha}}\right\}^2 - \frac{1}{32\pi\alpha^4}$$
$$- \frac{1}{8\alpha}\left\{\frac{1}{(1-\alpha)^2+\alpha^2} + \frac{1}{(2-\alpha)^2+\alpha^2} + \cdots \text{ to } 2\alpha \text{ terms}\right\}$$

In a similar manner we can show that, if α be a positive integer then

$$(17)\quad \frac{1}{1^4+4\alpha^4} + \frac{3}{3^4+4\alpha^4} + \frac{5}{5^4+4\alpha^4} + \cdots$$
$$= \frac{1}{4\alpha}\left\{\frac{1}{(1-\alpha)^2+\alpha^2} + \frac{1}{(2-\alpha)^2+\alpha^2} + \frac{1}{(3-\alpha)^2+\alpha^2} + \cdots \text{to } \alpha \text{ terms}\right\};$$

and (57)

$$(18)\quad \frac{1}{e^{\pi}+1}\cdot\frac{1}{1^4+4\alpha^4}+\frac{3}{e^{3\pi}+1}\cdot\frac{1}{3^4+4\alpha^4}+\frac{5}{e^{5\pi}+1}\cdot\frac{1}{5^4+4\alpha^4}+\cdots$$

$$=\frac{\pi}{32\alpha^2}\cdot\left\{\frac{e^{\pi\alpha}-(-1)^{\alpha}}{e^{\pi\alpha}+(-1)^{\alpha}}\right\}^2$$

$$-\frac{1}{8\alpha}\left\{\frac{1}{(1-\alpha)^4+\alpha^4}+\frac{1}{(3-\alpha)^4+\alpha^4}+\cdots \text{ to } \alpha \text{ terms}\right\}$$

It follows from (16) that, if 2α be a positive integer, then

$$\left\{\frac{1}{e^{2\pi}-1}-\frac{1}{e^{2\pi(1+2\alpha)}-1}\right\}+\sum_{n=1}^{n=\infty}\left\{\frac{1}{e^{2n\pi}-1}-\frac{1}{e^{2\pi(n+2\alpha)}-1}\right\}\frac{1}{(n+\alpha)^4+\alpha^2}$$

can be expressed in finite terms. Similarly from (18) we see that, if α be a positive integer, then

$$\sum_{n=1}^{n=\infty}\left\{\frac{1}{e^{(2n-1)\pi}+1}-\frac{1}{e^{(2\alpha+2n-1)\pi}+1}\right\}\frac{1}{(2n-1+\alpha)^4+\alpha^2}$$

can be expressed in finite terms. For example

$$\frac{1}{(1^4+2^2)(\sinh 3\pi-\sinh\pi)}+\frac{1}{(2^4+3^4)(\sinh 5\pi-\sinh\pi)}+\cdots$$

$$\cdots+\frac{1}{(3^4+4^4)(\sinh 7\pi-\sinh\pi)}+\cdots\cdots$$

$$=\left(\frac{1}{\pi}+\coth\pi-\tfrac{1}{2}\pi\tanh^2\tfrac{1}{2}\pi\right)\Big/\,2\sinh\pi\,.$$

(18) $$\frac{\sigma(1)}{1^2+x^2} - \frac{3\sigma(3)}{3^2+x^2} + \frac{5\sigma(5)}{5^2+x^2} - \cdots$$

$$= \frac{\pi}{4}\left\{ \operatorname{Sech}(\pi x/4) - \tfrac{1}{3}\operatorname{Sech}(\pi x/6) + \tfrac{1}{5}\operatorname{Sech}(\pi x/10) - \cdots \right\}$$

If $R(s) > 1$ and $R(s-\alpha) > 1$, it is evident that

(19) $$\zeta(s)\,\zeta(s-\alpha) = \frac{\sigma_\alpha(1)}{1^s} + \frac{\sigma_\alpha(2)}{2^s} + \frac{\sigma_\alpha(3)}{3^s} + \cdots$$

Similarly when $R(s) > 0$ and $R(s-\alpha) > 0$

(20) $$\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots\right)\left(\frac{1}{1^{s-\alpha}} - \frac{1}{3^{s-\alpha}} + \frac{1}{5^{s-\alpha}} - \cdots\right)$$

$$= \frac{\sigma_\alpha(1)}{1^s} - \frac{\sigma_\alpha(3)}{3^s} + \frac{\sigma_\alpha(5)}{5^s} - \frac{\sigma_\alpha(7)}{7^s} + \cdots$$

4. By the theory of residues it can be shown that

(21) $$\frac{1}{2\alpha} + \sum_{m=1}^{m=\infty}\left\{\frac{m\alpha\coth m\alpha}{n+m^2\alpha} + \frac{m\beta\coth m\beta}{n-m^2\beta}\right\} = \frac{\pi}{2}\cot\sqrt{(n\alpha)}\coth\sqrt{(n\beta)}$$

with the condition $\alpha\beta = \pi^2$. Similarly when $\alpha\beta = \pi^2/4$

(22) $$\sum_{m=0}^{m=\infty}(-1)^m\left\{\frac{(2m+1)\alpha\,\operatorname{Sech}(2m+1)\alpha}{(2m+1)^2\alpha + n} + \frac{(2m+1)\beta\,\operatorname{Sech}(2m+1)\beta}{(2m+1)^2\beta - n}\right\}$$

$$= \frac{\pi}{4}\operatorname{Sec}\sqrt{(n\alpha)}\operatorname{Sech}\sqrt{(n\beta)}.$$

Equating the coefficients of $\frac{1}{n}$ in both sides in (21) we have

(23) $$\alpha\left(\frac{1}{e^{2\alpha}-1} + \frac{2}{e^{4\alpha}-1} + \frac{3}{e^{6\alpha}-1} + \cdots\right)$$

$$+\beta\left(\frac{1}{e^{2\beta}-1} + \frac{2}{e^{4\beta}-1} + \frac{3}{e^{6\beta}-1} + \cdots\right)$$

$$= \frac{\alpha+\beta}{24} - \frac{1}{4}$$

with the condition $\alpha\beta = \pi^2$. In other words, if $\alpha\beta = \pi^2$ then

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(24) $$\alpha\left\{\sigma_1(1)e^{-2\alpha}+\sigma_1(2)e^{-4\alpha}+\sigma_1(3)e^{-6\alpha}+\cdots\right\}$$

$$+\beta\left\{\sigma_1(1)e^{-2\beta}+\sigma_1(2)e^{-4\beta}+\sigma_1(3)e^{-6\beta}+\cdots\right\}$$

$$=\frac{\alpha+\beta}{24}-\frac{1}{4}$$

The coefficient of $1/\pi^n$ in the expansion of the left-hand side of (21) must be 0, n being any positive integer ≥ 2, as there is no coefficient for $1/\pi^n$ in the right-hand side. That is, if $\alpha\beta=\pi^2$ and n any positive integer ≥ 2, then

(25) $$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\alpha}-1}+\frac{2^{2n-1}}{e^{4\alpha}-1}+\frac{3^{2n-1}}{e^{6\alpha}-1}+\cdots\right\}$$

$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\frac{1^{2n-1}}{e^{2\beta}-1}+\frac{2^{2n-1}}{e^{4\beta}-1}+\frac{3^{2n-1}}{e^{6\beta}-1}+\cdots\right\}$$

where $B_2=1/6$, $B_4=1/30$, $B_6=1/42$, ... the Bernoullian numbers.

(26) or $$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\sigma_{2n-1}(1)e^{-2\alpha}+\sigma_{2n-1}(2)e^{-4\alpha}+\sigma_{2n-1}(3)e^{-6\alpha}+\cdots\right\}$$

$$=(-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n+\sigma_{2n-1}(1)e^{-2\beta}+\sigma_{2n-1}(2)e^{-4\beta}+\sigma_{2n-1}(3)e^{-6\beta}+\cdots\right\}$$

As an example when $n=3$ and $\alpha=\beta=\pi$, we have

(27) $$\sigma_5(1)e^{-2\pi}+\sigma_5(2)e^{-4\pi}+\sigma_5(3)e^{-6\pi}+\cdots=\frac{1}{504}$$

Now equating the coefficients of π^n in both sides in (21) we can show that, if $\alpha\beta=\pi^2$ and n any positive integer ≥ 2, then

(28) $$(4\alpha)^{1-n}\left\{\tfrac{1}{2}\zeta(2n-1)+\frac{1^{1-2n}}{e^{2\alpha}-1}+\frac{2^{1-2n}}{e^{4\alpha}-1}+\frac{3^{1-2n}}{e^{6\alpha}-1}+\cdots\right\}$$

$$-(-4\beta)^{1-n}\left\{\tfrac{1}{2}\zeta(2n-1)+\frac{1^{1-2n}}{e^{2\beta}-1}+\frac{2^{1-2n}}{e^{4\beta}-1}+\frac{3^{1-2n}}{e^{6\beta}-1}+\cdots\right\}$$

$$= (-\alpha)^{1-n}\left\{\tfrac{1}{2}\left\{(2n-1) + \sigma_{1-2n}(1)\,e^{-2\alpha} + \sigma_{1-2n}(2)\,e^{-4\alpha} + \sigma_{1-2n}(3)\,e^{-6\alpha} + \cdots\right\}\right.$$

$$\left. - (-\beta)^{1-n}\left\{\tfrac{1}{2}\left\{(2n-1) + \sigma_{1-2n}(1)\,e^{-2\beta} + \sigma_{1-2n}(2)\,e^{-4\beta} + \sigma_{1-2n}(3)\,e^{-6\beta} + \cdots\right\}\right.\right.$$

$$= \frac{B_{2n}}{(2n)!}\left\{(-\alpha)^n + \beta^n\right\} + \pi^2\,\frac{B_2}{2!}\,\frac{B_{2n-2}}{(2n-2)!}\left\{(-\alpha)^{n-2} + \beta^{n-2}\right\}$$

$$- \pi^4\,\frac{B_4}{4!}\,\frac{B_{2n-4}}{(2n-4)!}\left\{(-\alpha)^{n-4} + \beta^{n-4}\right\} + \cdots\cdots\cdots$$

the last term being $-\pi^n\left(\frac{B_n}{n!}\right)^2(-1)^{n/2}$ or $\pi^{n-1}\frac{B_{n-1}}{(n-1)!}\frac{B_{n+1}}{(n+1)!}(-1)^{\frac{n+1}{2}}(\beta-\alpha)$ according as n is even or odd. Lastly equating the terms independent of n in both sides in (21) we have, when $\alpha\beta = \pi^2$,

$$(21)\quad \left\{\frac{1}{1(e^{2\alpha}-1)} + \frac{1}{2(e^{4\alpha}-1)} + \frac{1}{3(e^{6\alpha}-1)} + \cdots\right\}$$

$$- \left\{\frac{1}{1(e^{2\beta}-1)} + \frac{1}{2(e^{4\beta}-1)} + \frac{1}{3(e^{6\beta}-1)} + \cdots\right\}$$

$$= \left\{\sigma_{-1}(1)\,e^{-2\alpha} + \sigma_{-1}(2)\,e^{-4\alpha} + \sigma_{-1}(3)\,e^{-6\alpha} + \cdots\right\}$$

$$- \left\{\sigma_{-1}(1)\,e^{-2\beta} + \sigma_{-1}(2)\,e^{-4\beta} + \sigma_{-1}(3)\,e^{-6\beta} + \cdots\right\}$$

$$= \tfrac{1}{4}\log\left(\frac{\alpha}{\beta}\right) - \frac{\alpha-\beta}{12}.$$

Similarly from (22) we can obtain the following results. If $\alpha\beta = \pi^2/4$ and n is any positive integer excluding 0,

$$(30)\quad \alpha^n\left\{\frac{1^{2n-1}}{\cosh\alpha} - \frac{3^{2n-1}}{\cosh 3\alpha} + \frac{5^{2n-1}}{\cosh 5\alpha} - \cdots\right\}$$

$$+ (-\beta)^n\left\{\frac{1^{2n-1}}{\cosh\beta} - \frac{3^{2n-1}}{\cosh 3\beta} + \frac{5^{2n-1}}{\cosh 5\beta} - \cdots\right\} = 0$$

In other words

(31) $$\alpha^n \left\{ \sigma_{2n-1}(1)e^{-\alpha} - \sigma_{2n-1}(3)e^{-3\alpha} + \sigma_{2n-1}(5)e^{-5\alpha} - \right\}$$

$$+(-\beta)^n \left\{ \sigma_{2n-1}(1)e^{-\beta} - \sigma_{2n-1}(3)e^{-3\beta} + \sigma_{2n-1}(5)e^{-5\beta} - \right\} = 0$$

with the same condition as in (10). Again if $\alpha\beta = \pi^2/4$ and n is any positive integer excluding 0, then

(32) $$2\alpha^{1-n} \left\{ \frac{1^{1-2n}}{\cosh \alpha} - \frac{3^{1-2n}}{\cosh 3\alpha} + \frac{5^{1-2n}}{\cosh 5\alpha} - \cdots \right\}$$

$$+2(-\beta)^{1-n} \left\{ \frac{1^{1-2n}}{\cosh \beta} - \frac{3^{1-2n}}{\cosh 3\beta} + \frac{5^{1-2n}}{\cosh 5\beta} - \cdots \right\}$$

$$= 4\alpha^{1-n} \left\{ \sigma_{1-2n}(1)e^{-\alpha} - \sigma_{1-2n}(3)e^{-3\alpha} + \sigma_{1-2n}(5)e^{-5\alpha} - \cdots \right\}$$

$$+4(-\beta)^{1-n} \left\{ \sigma_{1-2n}(1)e^{-\beta} - \sigma_{1-2n}(3)e^{-3\beta} + \sigma_{1-2n}(5)e^{-5\beta} - \cdots \right\}$$

$$= \frac{\pi}{2} \cdot \frac{E_1}{0!} \cdot \frac{E_{2n-1}}{(2n-2)!} \left\{ (-\alpha)^{n-1} + \beta^{n-1} \right\} - \left(\frac{\pi}{2}\right)^3 \frac{E_3}{2!} \cdot \frac{E_{2n-3}}{(2n-4)!} \left\{ (-\alpha)^{n-3} + \beta^{n-3} \right\}$$

$$+ \left(\frac{\pi}{2}\right)^5 \frac{E_5}{4!} \cdot \frac{E_{2n-5}}{(2n-6)!} \left\{ (-\alpha)^{n-5} + \beta^{n-5} \right\} - \cdots\cdots$$

the last term being $(-1)^{\frac{n-1}{2}} \cdot \left(\frac{\pi}{2}\right)^n \left\{ \frac{E_n}{(n-1)!} \right\}^2$ or $(-1)^{\frac{n}{2}} \left(\frac{\pi}{2}\right)^{n-1} \frac{E_{n-1}}{(n-2)!} \cdot \frac{E_{n+1}}{n+1} (\alpha - \beta)$ according as n is odd or even, where $E_1 = 1$, $E_3 = 1$, $E_5 = 5$, ... the Eulerian numbers. As an example we have when $n = 1$,

(33) $$\left\{ \sigma_{-1}(1)e^{-\alpha} - \sigma_{-1}(3)e^{-3\alpha} + \sigma_{-1}(5)e^{-5\alpha} \cdots \right\}$$

$$+ \left\{ \sigma_{-1}(1)e^{-\beta} - \sigma_{-1}(3)e^{-3\beta} + \sigma_{-1}(5)e^{-5\beta} \cdots \right\} = \frac{\pi}{8}$$

with the condition $\alpha\beta = \pi^2/4$

Apparently 1st attempt at 7. pub (incomplete)

Eq(15) is not quite as published paper.

Add ref a 9th (68 v.)

(1)

Let

$$\Phi(x) = \int_0^x \frac{\tan^{-1} t}{t}\, dt$$

Then, changing t into $1/t$, it is easy to see that, if $x > 0$,

(1) $$\Phi(x) - \Phi\left(\tfrac{1}{x}\right) = \tfrac{1}{2}\pi \log x.$$

It is also clear that, if $-1 \le x \le 1$, then

(2) $$\Phi(x) = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots.$$

The following results can easily be proved by differentiating both sides with respect to x.

If $0 < x < \frac{1}{2}\pi$, then

(3) $$\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \frac{\sin 10x}{5^2} + \cdots$$

$$= \Phi(\tan x) - x \log \tan x.$$

As particular cases of (3) we have

(4) $$\sqrt{2}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \frac{1}{13^2} + \cdots\right) = \Phi(\sqrt{2}-1) + \frac{\pi}{8}\log(1+\sqrt{2}) + \frac{\pi^2}{16};$$

(5) $$\Phi(1) = \tfrac{3}{2}\Phi(2-\sqrt{3}) + \tfrac{1}{8}\pi \log(2+\sqrt{3}).$$

If $0 < x < \frac{1}{2}\pi$, then

(6) $$\frac{\cos x + \sin x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x + \sin^3 x}{3^2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos^5 x + \sin^5 x}{5^2} + \cdots$$

$$= \Phi(\tan x) + \tfrac{1}{2}\pi \log(2\cos x);$$

and

(7) $$\frac{\sin x}{1^2}\cos x + \frac{\sin 2x}{2^2}\cos^2 x + \frac{\sin 3x}{3^2}\cos^3 x + \cdots$$

$$= \Phi(\tan x) + \tfrac{1}{2}\pi \log \cos x - x \log \sin x.$$

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and l is greater than or equal to either $|(1-\alpha)\sin x|$ or $|(1-1/\alpha)\cos x|$, then

$$\left(1-\frac{1}{3^2}\right)^{-3}\left(1-\frac{1}{5^2}\right)^{5}\left(1-\frac{1}{7^2}\right)^{-7}\left(1-\frac{1}{9^2}\right)^{9} \cdots = \frac{\pi}{8} e^{\frac{4}{\pi}\Phi(1)}$$

$$\log\left(1+\frac{64x^6}{1^2}\right) - 3\log\left(1+\frac{64x^6}{3^2}\right) + 5\log\left(1+\frac{64x^6}{5^2}\right) - \cdots$$

$$= \frac{8}{\pi}\Phi(1) - 2x\log\left(\frac{\cosh\pi x + i\sin\pi x}{\cosh\pi x - i\sin\pi x}\right) - 4x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$

$$- \frac{8}{\pi}\int \frac{\cos\pi x\, e^{-\pi x}}{1^2} - \frac{\cos 3\pi x\, e^{-3\pi x}}{3^2} + \cdots \Big\}$$

$$\tan^{-1}\frac{8x^3}{1^2} - 3\tan^{-1}\frac{8x^3}{3^2} + 5\tan^{-1}\frac{8x^3}{5^2} - \cdots$$

$$2\log\left(\frac{\cosh\pi x + i\sin\pi x}{\cosh\pi x - i\sin\pi x}\right) - 2x\tan^{-1}\left(\frac{\cos\pi x}{\sinh\pi x}\right)$$

$$+ \frac{8}{\pi}\Big\{ \frac{\sin\pi x\, e^{-\pi x}}{1^2} - \frac{\sin 3\pi x\, e^{-3\pi x}}{3^2} + \frac{\sin 5\pi x\, e^{-5\pi x}}{5^2} - \cdots \Big\}$$

n positive odd integer $\left(1+\frac{4n^3}{1^2}\right)\left(1+\frac{4n^3}{3^2}\right)^{-3}\left(1+\frac{4n^3}{5^2}\right)^{5} \cdots$

$$= \left(\frac{1-e^{-4\pi n}}{1+e^{-4\pi n}}\right)^{2n(-1)^{\frac{1}{2}(n-1)}} e^{\frac{8}{\pi}\Phi(1)},$$

n any even integer $\left(1+\frac{4n^3}{1^2}\right)\left(1+\frac{4n^3}{3^2}\right)^{-3}\left(1+\frac{4n^3}{5^2}\right)^{5} \cdots$

$$= \exp\Big\{ \frac{8}{\pi}\Phi(1) - \frac{8}{\pi}(-1)^{\frac{1}{2}n}\Big(\Phi(e^{-4\pi n}) + 4\pi n \tan^{-1} e^{-4\pi n} \Big) \Big\}$$

n positive odd integer $\tan^{-1}\frac{2n^3}{1^2} - 3\tan^{-1}\frac{2n^3}{3^2} + \cdots$

$$\frac{4}{\pi}(-1)^{\frac{1}{2}(n-1)}\Big\{ \frac{\pi n}{4}\log\left(\frac{1+e^{-4\pi n}}{1-e^{-4\pi n}}\right) + e^{-4\pi n} - \frac{e^{-8\pi n}}{3^2} + \cdots \Big\}$$

positive even integer $\tan^{-1}\frac{2n^3}{1^2} - 3\tan^{-1}\frac{2n^3}{3^2} + \cdots$

$$= n(-1)^{\frac{1}{2}n-1}\tan^{-1} e^{-4\pi n}.$$

(8) $\frac{\sin x}{1^2}(1-\frac{1}{\alpha})\cos x + \frac{\sin 2x}{2^2}(1-\frac{1}{\alpha})^2\cos^2 x + \cdots\cdots$

$= \frac{\sin(x+\frac{1}{2}\pi)}{1^2}(1-\alpha)\sin x + \frac{\sin 2(x+\frac{1}{2}\pi)}{2^2}(1-\alpha)^2\sin^2 x + \cdots$

$= \Phi(\tan x) - \Phi(\alpha\tan x) + x\log\alpha.$

If $-1 < x < 1$, then

(9) $\log(1-\frac{x^2}{1^2}) - 3\log(1-\frac{x^2}{3^2}) + 5\log(1-\frac{x^2}{5^2}) - \cdots$

$= \frac{4}{\pi}\left\{\Phi(1) - \Phi(\tan\tfrac{1}{4}\pi(1+x))\right\} + \log\tan\tfrac{1}{4}\pi(1-x).$

As an example we have

(10) $(1-\frac{4}{3^2})(1-\frac{4}{9^2})^{-3}(1-\frac{4}{15^2})^{5}(1-\frac{4}{21^2})^{-7}\cdots$

$= \frac{e^{\frac{4}{3\pi}\Phi(1)}}{(2+\sqrt{3})^{8/3}}.$

If x is real, then

(11) $\log(1+\frac{x^2}{1^2}) - 3\log(1+\frac{x^2}{3^2}) + 5\log(1+\frac{x^2}{5^2}) - \cdots$

$= \frac{4}{\pi}\left\{\Phi(1) - \Phi(e^{-\frac{1}{2}\pi x})\right\} - 2x\tan^{-1}e^{-\frac{1}{2}\pi x}.$

For example

(12) $\left(1+\left\{\frac{2}{\pi}\log(2+\sqrt{3})\right\}^2\right)\left(1+\left\{\frac{2}{3\pi}\log(2+\sqrt{3})\right\}^2\right)^{-3}\left(1+\left\{\frac{2}{5\pi}\log(2+\sqrt{3})\right\}^2\right)^{5}\cdots$

$= e^{\frac{4}{3\pi}\Phi(1)}$

It follows from (9) and (11) that, if $-1 < \beta < 1$ and $\frac{1}{2}\pi\alpha = \log\tan\frac{1}{4}\pi(1+\beta)$, then

(13) $\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\left(\frac{7^2-\beta^2}{7^2+\alpha^2}\right)^7\cdots\cdots$

$= e^{\frac{1}{2}\pi\alpha\beta}.$

Add [illegible] 18 (67) (3)

By the theory of residues it can be shown that, if α and β are positive and $\alpha\beta = \pi^2$, then

$$(14) \quad \frac{1}{1^2(e^{\alpha}-1)} - \frac{1}{3^2(e^{3\alpha}-1)} + \frac{1}{5^2(e^{5\alpha}-1)} - \cdots$$

$$+ \frac{\pi}{4\beta}\left\{\frac{1}{1^2(e^{\beta}+e^{-\beta})} + \frac{1}{2^2(e^{2\beta}+e^{-2\beta})} + \frac{1}{3^2(e^{3\beta}+e^{-3\beta})} + \cdots\right\}$$

$$= \frac{\pi}{16}\left(\frac{\alpha}{3} + \frac{\beta}{2}\right) - \frac{1}{2}\Phi(1).$$

Putting $\alpha = \beta = \pi$ in (14) we can easily calculate the approximately value of $\Phi(1)$. Thus

$$\Phi(1) = .915\,965\,594\,2$$

approximately.

If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then

$$(15) \quad \sin x + \frac{2}{3}\cdot\frac{\sin^3 x}{3} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin^5 x}{5} + \cdots$$

$$= 2\,\Phi(\tan\tfrac{1}{2}x).$$

1. If $d(N)$ be the no of divisors of N, then

$$C^{\frac{N}{\log N}(1-\varepsilon)} < d(N) < C^{\frac{N}{\log N}(1+\varepsilon)}$$

old & well-known

where $C = (1+1)\sqrt{1+\frac{1}{2}}\sqrt[3]{1+\frac{1}{3}}\sqrt[4]{1+\frac{1}{4}}\sqrt[5]{1+\frac{1}{5}}\cdots\cdots$

2. If $S(N)$ be the no of ways in which N can be expressed as the sum of 2 Sq^{s}, then the max order of $S(N)$

$$= \sqrt{\text{max order of } d(N^2+aN+b)} \cdot e^{O(\log N)^{\frac{1}{2}+\varepsilon}}$$

3. No of the form x^n+y^n less than a given no N (where $n>3$)

$$= \frac{N^{\frac{2}{n}}}{4n}\cdot\frac{\left\{\Gamma\left(\frac{1}{2n}\right)\right\}^2}{\Gamma\left(\frac{1}{n}\right)} + O\left(N^{\frac{1}{n}+\varepsilon}\right)$$

[assuming the equation $x^n+y^n = u^n+v^n$ (x and y being different from u and v) has not got an infinite no of solutions]

4. The coeff^{t} of x^m in $\frac{1}{1-2x+2x^4-x^9+\cdots}$ is nearly

$$\frac{1}{4m}\left\{\cosh(\pi\sqrt{m}) - \frac{\sinh(\pi\sqrt{m})}{\pi\sqrt{m}}\right\}$$

$$(1-2x+2x^4-\cdots)^{-1} = 1+2x+4x^2+8x^3+14x^4+24x^5+40x^6$$
$$+ 70x^7+100x^8+154x^9+232x^{10}+344x^{11}+504x^{12}+728x^{13}+$$
$$1040x^{14}+1472x^{15}+2062x^{16}+2864x^{17}+3948x^{18}+5400x^{19}+$$
$$7336x^{20}+9904x^{21}+13288x^{22}+17728x^{23}+23528x^{24}+$$
$$31025x^{25}+40824x^{26}+53408x^{27}+69568x^{28}+90248x^{29}$$
$$+116624x^{30}+150144x^{31}+192612x^{32}+246256x^{33}+$$
$$315508x^{34}+398640x^{35}+504886x^{36}+\cdots\cdots\cdots$$

$$\int_0^\infty \frac{1}{\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots\left\{1+\left(\frac{x}{a+n}\right)^2\right\}} \cdot \frac{1}{\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\cdots\left\{1+\left(\frac{x}{a+n}\right)^2\right\}} \cdot \frac{dx^2}{1+x^{2n}}$$

$$= \frac{2\sqrt{\pi}\cdot\dfrac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+\frac{b}{2}\right)\Gamma\left(1+\frac{a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)}}{a+b+1} + \frac{1}{a+b+3} + \frac{2(a+1)(a+2)(a+3)(a+4)\pi^2}{a+b+5} + \cdots\cdots$$

$a+1$ and $b+1$ being positive.

If $n=1$, the continued fraction can be shown to be equal to the series

$$\frac{1}{a+b+1}\left(1 - A_1 + A_1A_2 - A_1A_2A_3 + \cdots\cdots\right)$$

where $A_k = \dfrac{(a+b)(b+k)}{(a+1+k)(b+1+k)} \dfrac{\tan^2\frac{k\pi}{2}}{\tan^2\frac{k\pi}{2}}$.

Add to p. 34 '8 (69)

(69)

$$\sum a_n x^n = x^0 \left\{(1-x^4)(1-x^9)(1-x^{19})\cdots\right\}^{18}$$

Add Hu a 94 18 (70)

156 $$\sum \frac{a_n}{n^s} = \frac{1}{1-7p\cdot 3^{-s}+3^{p-2s}} \cdot \frac{1}{1+5\cdot 10\cdot 5^{-s}+5^{p-2s}}$$

$$\times \frac{1}{1+1404\cdot 7^{-s}+7^{p-2s}} \times \cdots$$

$$- \frac{1}{1+7p\cdot 3^{-s}+3^{p-2s}} \cdot \frac{1}{1+5\cdot 10\cdot 5^{-s}+5^{p-2s}}$$

$$\times \frac{1}{1-1404\cdot 7^{-s}+7^{p-2s}} \times \cdots$$

Presumably there are analogous results for $\sum \frac{a_n}{n^s}$ where $\sum a_n x^n$ is any of the functions

$$x^5\left\{(1-x^{12})(1-x^{24})(1-x^{36})\cdots\right\}^{10},$$

$$x^7\left\{(1-x^{12})(1-x^{24})(1-x^{36})\cdots\right\}^{14},$$

$$x^5\left\{(1-x^6)(1-x^{12})(1-x^{18})\cdots\right\}^{20},$$

$$x^{11}\left\{(1-x^{12})(1-x^{24})(1-x^{36})\cdots\right\}^{22}.$$

See 'certain arithmetical'

$$F(x) = x^2\left\{(1-x^3)(1-x^6)(1-x^9)\cdots\right\}^{16}$$

$$F_1(x) = x\left\{(1-x^3)(1-x^6)(1-x^9)\cdots\right\}^{8}\left\{1+240\left(\frac{1^3x^3}{1-x^3}+\frac{2^3x^6}{1-x^6}+\ldots\right)\right\}$$

$$\sum A_n x^n = F_1(x), \qquad \sum a_n x^n = F(x).$$

$$\sum \frac{A_n + 6a_n\sqrt{10}}{n^s} = \frac{1}{1-6\sqrt{10}\cdot 2^{-s}+2^{7-2s}}\cdot\frac{1}{1+96\sqrt{10}\cdot 5^{-s}+5^{7-2s}}$$

$$\times\frac{1}{1-260\cdot 7^{-s}+7^{7-2s}}\cdot\frac{1}{1+1920\sqrt{10}\cdot 11^{-s}+11^{7-2s}}\cdots\cdots$$

$$\sum \frac{A_n - 6a_n\sqrt{10}}{n^s} = \frac{1}{1+6\sqrt{10}\cdot 2^{-s}+2^{7-2s}}\cdot\frac{1}{1-96\sqrt{10}\cdot 5^{-s}+5^{7-2s}}$$

$$\times\frac{1}{1-260\cdot 7^{-s}+7^{7-2s}}\cdot\frac{1}{1-1920\sqrt{10}\cdot 11^{-s}+11^{7-2s}}\cdots$$

See "A Theorem in Cl. Anal."

$$(1-v^{1^3})(1-v^{2^3})(1-v^{3^3})(1-v^{4^3})\ldots\ldots$$

Added Aug 9th 1913.

$$(2\pi x)^{\frac{1}{2}} e^{\pi x \zeta(-3)} \left(1-e^{-2\pi x\cdot 1^5}\right)\left(1-e^{-2\pi x\cdot 2^5}\right)\left(1-e^{-2\pi x\cdot 3^5}\right)\ldots.$$

$$= (2\pi)^{\frac{1}{2}5} e^{\frac{\pi\zeta(-\frac{1}{5})}{x^{\frac{1}{5}}\sin\frac{\pi}{10}}} \prod_{\lambda=1}^{\lambda=\infty}\left\{1 - 2e^{-2\pi(\frac{\lambda}{x})^{\frac{1}{5}}\sin\frac{\pi}{10}}\cos\left(2\pi(\tfrac{\lambda}{x})^{\frac{1}{5}}\cos\tfrac{\pi}{10}\right) + e^{-4\pi(\frac{\lambda}{x})^{\frac{1}{5}}\sin\frac{\pi}{10}}\right\}^{\frac{1}{2}}$$

$n = 1, 3, 5, 7, \ldots\ldots 2s-1.$

$s=1,$ $\quad \sqrt{x}\, e^{-\frac{\pi x}{12}}(1-e^{-2\pi x})(1-e^{-4\pi x})(1-e^{-6\pi x})\ldots. = e^{-\frac{\pi}{12x}}(1-e^{-\frac{2\pi}{x}})(1-e^{-\frac{4\pi}{x}})\ldots.$

$s=2,$ $\quad \frac{1}{2}\sqrt{(\frac{x}{\pi})}\, e^{-\frac{2\pi}{\sqrt{x}}\zeta(-\frac{1}{2})}(1-e^{-\pi x})(1-e^{-4\pi x})(1-e^{-9\pi x})(1-e^{-16\pi x})\ldots$

$$= \prod_{\lambda=1}^{\lambda=\infty}\left\{1 - 2e^{-2\pi\sqrt{(\frac{\lambda}{x})}}\cos\left(2\pi\sqrt{\tfrac{\lambda}{x}}\right) + e^{-4\pi\sqrt{(\frac{\lambda}{x})}}\right\}.$$

$s=3,$ $\quad \sqrt{x}\, e^{\frac{\pi x}{120}} - \frac{2\pi\zeta(-\frac{4}{5})}{\sqrt{x}}(1-e^{-2\pi x})(1-e^{-16\pi x})(1-e^{-54\pi x})\ldots.$

$$= 2\pi \prod_{\lambda=1}^{\lambda=\infty}\left\{\left(1-e^{-\pi\sqrt[3]{\frac{\lambda}{x}}}\right)\left(1-2e^{-\pi\sqrt[3]{\frac{\lambda}{x}}}\cos\left(\pi/3\sqrt[3]{\tfrac{\lambda}{x}}\right)+e^{-2\pi\sqrt[3]{\frac{\lambda}{x}}}\right)\right\}$$

$$\sum c_n x^n = \frac{1}{1 - x + x^3 - x^6 + \dots}$$

$$c_n = \frac{d}{dn} \frac{\cosh \pi\sqrt{n} - 1}{2\pi\sqrt{n}} + \sqrt{3}\, 2\cos\left(\frac{2n\pi}{3} - \frac{\pi}{6}\right) \frac{d}{dn} \frac{\cosh \frac{\pi}{3}\sqrt{n} - 1}{2\pi\sqrt{n}} + \dots$$

$$\sum c_n x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\dots} = (1+x)(1+x^3)(1+x^5)\dots$$

$$\therefore \sqrt{2} = \frac{d}{dn} J_0\left\{i\pi\sqrt{\tfrac{1}{3}(n+\tfrac{1}{24})}\right\} + 2\cos\left(\frac{2n\pi}{3} - \frac{\pi}{9}\right) \frac{d}{dn} J_0\left\{\frac{i\pi}{3}\sqrt{\tfrac{1}{3}(n+\tfrac{1}{24})}\right\} + \dots$$

$$\sum c_n x^n = (1+x)(1+x^3)(1+x^5)\dots$$

$$= \frac{d}{dn} J_0\left\{i\pi\sqrt{\tfrac{1}{6}(n-\tfrac{1}{24})}\right\} + 2\cos\left(\frac{2n\pi}{3} - \frac{2\pi}{9}\right) \frac{d}{dn} J_0\left\{\frac{i\pi}{3}\sqrt{\tfrac{1}{6}(n-\tfrac{1}{24})}\right\} + \dots$$

$$\sum c_n x^n = \frac{1}{(1-ax)(1-ax^3)(1-ax^5)\dots}, \quad 0 < a < 1.$$

$$c_n \sim \sqrt{1-a}\, \frac{(hn)^{\frac{1}{4}}}{2n\sqrt{\pi}}\, e^{2\sqrt{hn}},$$

$$h = \frac{a}{1^2} + \frac{a^2}{2^2} + \frac{a^3}{3^2} + \frac{a^4}{4^2} + \dots$$

$$(1 - e^{-2\pi x \cdot 1^5})(1 - e^{-4\pi x \cdot 2^3})(1 - e^{-6\pi x \cdot 3^5}) \dots$$

$$= \frac{(2\pi)^{\frac{1}{2}(s-1)}}{\sqrt{x}} \cdot \exp\left\{\frac{\pi f(-\frac{1}{3})}{x^{\frac{1}{3}} \sin\frac{\pi}{2s}} - \pi x f(-s)\right\} \times$$

$$\prod_{\lambda} \prod_{\nu=1}^{\nu=\infty} \left\{1 - 2e^{-2\pi\left(\frac{\nu}{2s}\right)^{\frac{1}{s}} \sin\frac{\pi\lambda}{2s}} \cos\left(2\pi\left(\frac{\nu}{2s}\right)^{\frac{1}{s}} \cos\frac{\pi\lambda}{2s}\right) + e^{-4\pi\left(\frac{\nu}{2s}\right)^{\frac{1}{s}} \sin\frac{\pi\lambda}{2s}}\right\}$$

$\lambda = 1, 3, 5, 7, \dots, 2s-1.$

(1) If $\alpha\beta = 4\pi^3$, then

$$\frac{1}{e^{\alpha}-1} + \frac{1}{e^{4\alpha}-1} + \frac{1}{e^{9\alpha}-1} + \frac{1}{e^{16\alpha}-1} + \cdots$$

$$= \frac{\pi^2}{6\alpha} + \frac{1}{4} + \frac{\sqrt{\beta}}{4\pi}\left\{ f\left(\tfrac{1}{2}\right) + \sum_{n=1}^{n=\infty} \frac{\cos\sqrt{(n\beta)} - \sin\sqrt{(n\beta)} - e^{-\sqrt{(n\beta)}}}{\cosh\sqrt{(n\beta)} - \cos\sqrt{(n\beta)}} \right\}.$$

In other words, if

$$\frac{x}{1-x} + \frac{x^4}{1-x^4} + \frac{x^9}{1-x^9} + \cdots = \sum \phi(n)\, x^n$$

and $$\frac{x}{1-x} + \frac{2x^4}{1-x^4} + \frac{3x^9}{1-x^9} + \cdots = \sum \psi(n)\, x^n,$$

then

$$\phi(1)e^{-\alpha} + \phi(2)e^{-2\alpha} + \phi(3)e^{-3\alpha} + \cdots$$

$$= \frac{\pi^2}{6\alpha} + \frac{\sqrt{\beta}}{2\pi}\left\{ \tfrac{1}{2} f\left(\tfrac{1}{2}\right) + \frac{\psi(1)}{\sqrt{1}} e^{-\sqrt{\beta}}(\cos\sqrt{\beta} - \sin\sqrt{\beta}) + \frac{\psi(2)}{\sqrt{2}} e^{-\sqrt{(2\beta)}}(\cos\sqrt{(2\beta)} - \sin\sqrt{(2\beta)}) \right.$$

$$\left. + \frac{\psi(3)}{\sqrt{3}} e^{-\sqrt{(3\beta)}}(\cos\sqrt{(3\beta)} - \sin\sqrt{(3\beta)}) + \cdots \right\}.$$

with the condition $\alpha\beta = 4\pi^3$

$$\frac{1^n}{e^{1^2\alpha}-1} + \frac{2^n}{e^{2^2\alpha}-1} + \frac{3^n}{e^{3^2\alpha}-1} + \cdots$$

where n is a positive integer and $n-5$ is an even integer.

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})$$

$$\frac{1}{(x-1)(x^2-1)(x^4-1)\cdots(x^{2^{n-1}}-1)}$$

$$\frac{x-1}{x^{2^n}-1},\ \frac{x^2-1}{x^{2^n}-1},\ \frac{x^4-1}{x^{2^n}-1},\ \cdots,\ \frac{x^{2^{n-1}}-1}{x^{2^n}-1}$$

$$\phi(x) = 1 + x + x^2 + x^3 + \cdots$$

$$\phi^2(x) + \phi(x^2) = \phi^2(x^2)$$

Add in a 94 [13(75)] [15]

$$\frac{1^{1-s}}{e^{2\pi}-1} + \frac{2^{1-s}}{e^{4\pi}-1} + \frac{3^{1-s}}{e^{6\pi}-1} + \cdots$$

$$= S_s - \tfrac{1}{2}\zeta(s-1) + \tfrac{3}{4\pi}\zeta(s) + \tfrac{\pi}{12}\zeta(s-2).$$

$S_{-4m} = 0$, $S_0 = \frac{1}{4\pi}$, $S_4 = -\frac{\pi^3}{360}$, $S_8 = 0$, $S_{12} = -\frac{\pi^{11}}{232186500}$

$S_{16} = -\frac{\pi^{15}}{2147087250}$, $S_{20} = -\frac{191\pi^{19}}{398240480137600}$

$S_{24} = \frac{907\pi^{23}}{184177717143590625}$.

Compare G H H A 7th of R
(Jour. L)

$0 < \theta < 1.$

$$\left[\frac{x}{1}\right]\sin 2\pi\theta + \left[\frac{x}{2}\right]\sin 4\pi\theta + \left[\frac{x}{3}\right]\sin 6\pi\theta + \left[\frac{x}{4}\right]\sin 8\pi\theta + \cdots$$

$$= \pi x\left(\tfrac{1}{2}-\theta\right) - \tfrac{1}{4}\cot\pi\theta + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{J_1\left(4\pi\sqrt{\lambda\theta x}\right)}{\sqrt{\lambda\theta}} - \frac{J_1\left(4\pi\sqrt{\lambda(1-\theta)x}\right)}{\sqrt{\lambda(1-\theta)}} + \right.$$

$$\left. \frac{J_1\left(4\pi\sqrt{\lambda(1+\theta)x}\right)}{\sqrt{\lambda(1+\theta)}} - \frac{J_1\left(4\pi\sqrt{\lambda(2-\theta)x}\right)}{\sqrt{\lambda(2-\theta)}} + \frac{J_1\left(4\pi\sqrt{\lambda(2+\theta)x}\right)}{\sqrt{\lambda(2+\theta)}} - \cdots\cdots\right\}$$

where $[x]$ denotes the greatest integer in x if x is not an integer and $x-\frac{1}{2}$ if x is an integer.

$$\left[\frac{x}{1}\right]\cos 2\pi\theta + \left[\frac{x}{2}\right]\cos 4\pi\theta + \left[\frac{x}{3}\right]\cos 6\pi\theta + \left[\frac{x}{4}\right]\cos 8\pi\theta + \cdots$$

$$= -x\log(2\sin\pi\theta) + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{I_1\left(4\pi\sqrt{\lambda\theta x}\right)}{\sqrt{\lambda\theta}} + \frac{I_1\left(4\pi\sqrt{\lambda(1-\theta)x}\right)}{\sqrt{\lambda(1-\theta)}} + \right.$$

$$\left. \frac{I_1\left(4\pi\sqrt{\lambda(1+\theta)x}\right)}{\sqrt{\lambda(1+\theta)}} + \frac{I_1\left(4\pi\sqrt{\lambda(2-\theta)x}\right)}{\sqrt{\lambda(2-\theta)}} + \frac{I_1\left(4\pi\sqrt{\lambda(2+\theta)x}\right)}{\sqrt{\lambda(2+\theta)}} + \cdots\cdots\right\}.$$

where

$$I_1(x) = H_1(x) - Y_1(x).$$

$\sigma_s(n)$ is the sum of the s^{th} powers of the divisors of n.

$$\Gamma\left(s+\tfrac{1}{2}\right)\left\{\frac{\zeta(1-s)}{\left(s-\tfrac{1}{2}\right)x^{s-\frac{1}{2}}}+\frac{\zeta(-s)\tan\frac{1}{2}\pi s}{2x^{s+\frac{1}{2}}}+\sum_{\lambda=1}^{\lambda=\infty}\frac{\sigma_s(\lambda)}{2\lambda^{\frac{1}{2}}}\left[(x-i\lambda)^{-s-\frac{1}{2}}-(x+i\lambda)^{-s-\frac{1}{2}}\right]\right\}$$

$$=(2\pi)^s\left\{\frac{\zeta(1-s)}{2\sqrt{\pi x}}-2\pi\sqrt{\pi x}\,\zeta(-s)\tan\tfrac{1}{2}\pi s+\sqrt{\pi}\sum_{\lambda=1}^{\lambda=\infty}\frac{\sigma_s(\lambda)}{\sqrt{\lambda}}e^{-2\pi\sqrt{\lambda x}}\sin\left(\tfrac{\pi}{4}+2\pi\sqrt{\lambda x}\right)\right\}$$

If $\alpha\beta=4\pi^2$, then

$$\alpha^{\frac{s+1}{2}}\left\{\tfrac{1}{2}\zeta(1-s)+\tfrac{1}{2}\zeta(-s)\tan\tfrac{1}{2}\pi s+\sigma_s(1)\sin\alpha+\sigma_s(2)\sin 2\alpha+\ldots\right\}$$

$$=\beta^{\frac{s+1}{2}}\left\{\tfrac{1}{2}\zeta(1-s)+\tfrac{1}{2}\zeta(s)\tan\tfrac{1}{2}\pi s+\sigma_s(1)\sin\beta+\sigma_s(2)\sin 2\beta+\ldots\right\}$$

See Hardy-Littlewood [illegible] (78)

(numbers exceeding 1) [illegible] (78)

$p(x)$ is the no. of ways in which no p [illegible] summands does not exceed k.

$$p(x) \sim \frac{x}{\log x}\left\{1 + \frac{\log_2 x}{1!} + \frac{(\log_2 x)^2}{2!} + \cdots + \frac{(\log_2 x)^{[k]}}{[k]!}\right\}.$$

This is true when k is infinite.

Is this true when k is a function of x?

$p(x)$ is the no. of ways of the form

$$p^{a_1} p^{a_2} p^{a_3} \cdots p^{a_r} \qquad p^a p \qquad p \le x^\varepsilon$$

not exceeding x.

$\frac{1}{2} \le \varepsilon \le 1$. $\quad \phi(x) \sim x\left\{1 - \int_\varepsilon^1 \frac{d\lambda_0}{\lambda_0}\right\}$.

$\frac{1}{3} \le \varepsilon \le \frac{1}{2}$. $\quad \phi(x) \sim x\left\{1 - \int_\varepsilon^1 \frac{d\lambda_0}{\lambda_0} + \int_\varepsilon^{\frac{1}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0}\right\}$

$\frac{1}{4} \le \varepsilon \le \frac{1}{3}$ $\quad \phi(x) \sim x\left\{1 - \int_\varepsilon^1 \frac{d\lambda_0}{\lambda_0} + \int_\varepsilon^{\frac{1}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0} - \int_\varepsilon^{\frac{1}{3}} \frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0}\right\}$

$\frac{1}{5} \le \varepsilon \le \frac{1}{4}$ $\quad \phi(x) \sim x\left\{1 - \int_\varepsilon^1 \frac{d\lambda_0}{\lambda_0} + \int_\varepsilon^{\frac{1}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0} - \int_\varepsilon^{\frac{1}{3}} \frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0} + \int_\varepsilon^{\frac{1}{4}} \frac{d\lambda_3}{\lambda_3}\int_{\lambda_3}^{\frac{1-\lambda_3}{2}} \frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_3}{2}} \frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1} \frac{d\lambda_0}{\lambda_0}\right\}$

and so on.

1. $x + (x-1) + (x-2) + \cdots = \frac{x^2}{2} + \frac{x}{2} + \begin{cases} + \frac{1}{8} \\ + 0 \end{cases}$

2. $x^2 + (x-1)^2 + (x-2)^2 + \cdots = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} + \begin{cases} + \frac{1}{24}\sqrt{3} \\ - \frac{1}{24}\sqrt{3} \end{cases}$

3. $x^3 + (x-1)^3 + (x-2)^3 + \cdots = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4} + \begin{cases} + 0 \\ - \frac{1}{64} \end{cases}$

4. $x^4 + (x-1)^4 + (x-2)^4 + \cdots = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} + \begin{cases} + \frac{1}{900}\sqrt{15+4\sqrt{5}} \\ - \frac{1}{900}\sqrt{15+4\sqrt{5}} \end{cases}$

5. $x^5 + (x-1)^5 + (x-2)^5 + \cdots = \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12} + \begin{cases} + \frac{1}{192} \\ + 0 \end{cases}$

6. $x^6 + (x-1)^6 + (x-2)^6 + \cdots = \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42} + \begin{cases} + \\ - \end{cases}$

$$\frac{1}{2352}\sqrt{\left\{(29\tfrac{8}{3} + 11\sqrt{2})^{3}\sqrt{21} - 7\sqrt{2} + (29\tfrac{8}{3} - 11\sqrt{2})^{3}\sqrt{21} + 7\sqrt{2} - 11\right\}}.$$

7. $x^7 + (x-1)^7 + (x-2)^7 + \cdots = \frac{x^8}{8} + \frac{x^7}{2} + \frac{7x^6}{12} - \frac{7x^4}{24} + \frac{x^2}{12} + \begin{cases} + 0 \\ - \frac{17}{2048} \end{cases}$

[illegible]

If $x \geqslant 0$, $\Gamma(1+x) = \sqrt{\pi} \cdot \left(\frac{x}{e}\right)^x \left\{8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right\}^{1/6}$

where θ_x is a positive proper fraction.

$$\theta_0 = \frac{30}{\pi^3} = .9675$$

$\theta_{1/12} = .8071$	$\theta_{7/12} = .3058$
$\theta_{2/12} = .6160$	$\theta_{8/12} = .3014$
$\theta_{3/12} = .4867$	$\theta_{9/12} = .3041$
$\theta_{4/12} = .4029$	$\theta_{10/12} = .3118$
$\theta_{5/12} = .3509$	$\theta_{11/12} = .3227$
$\theta_{6/12} = .3207$	$\theta_1 = .3359$

$$\theta_\infty = 1.$$

$$\sqrt{\pi} \cdot \left(\frac{x}{e}\right)^x \left\{8x^3 + 4x^2 + x + \frac{1}{100}\right\}^{1/6} < \Gamma(1+x) < \sqrt{\pi} \cdot \left(\frac{x}{e}\right)^x \left\{8x^3 + 4x^2 + x + \frac{1}{30}\right\}^{1/6}$$

0	31
$\frac{1}{2}$	94
1	89
$1\frac{1}{2}$	

$$(x+a)^5 - (x^5+b^5) = A(x^5+p)^5 + B(x+q)^5 + C(x^5+r)^5 \quad (8)$$

$$\frac{A}{1-\beta z} + \frac{B}{1-\delta z} + \frac{C}{1-\lambda z} = \frac{\alpha+\beta z+\gamma z^2}{1+\delta z+\varepsilon z^2+\phi z^3}.$$

$$\alpha = 5(a-b), \quad \beta = -2\frac{a-b}{a+2}(2a^2+3ab+4b^2), \quad \gamma = 2(a^3-b^3),$$

$$\delta = -2\frac{a^2+ab+b^2}{a+b}, \quad \varepsilon = a^2+ab+b^2, \quad \phi = -\frac{a^4+6a^3b+6ab^3+5ab^2+b^4}{10(a+b)}$$

If $z = \frac{1}{N}$, $N = \frac{1}{2}(a+b) + \frac{1}{2}M(a-b)$, then the equation

$$1+\delta z+\varepsilon z^2+\phi z^3 = 0$$

is equivalent to

$$5(a+b)(M^3-M) = (a^2-b^2)(5M^2-1).$$

$$x\left\{(x+a)^5+(x+b)^5\right\} = A(x+p)^5 + B(x+q)^5 + Cx^5,$$

where $p = \dfrac{a^3+b^3}{a^2+b^2-(a-b)\sqrt{3ab}}$, $q = -\dfrac{a^3+b^3}{a^2+b^2+(a-b)\sqrt{3ab}}$

$$B = \frac{\left(a^2+b^2+a-b\sqrt{3ab}\right)^5}{8(a-b)\sqrt{3ab}\,(a^2+b^2)^2}, \quad A = -\frac{\left(a^2+b^2-a-b\sqrt{3ab}\right)^5}{8(a-b)\sqrt{3ab}\,(a^2+b^2)^2}$$

$$C = \frac{(a^2-b^2)(a-b)^5}{(a^2+b^2)^2}.$$

$$\left\{x^5(n^2+3)^2(n^4+42n^2+9)^2 + x(n^2-3)(n^2+6n+3)\right\}^5$$

$$-\left\{x^5(n^2+3)^2(n^4+42n^2+9)^2 + x(n^2-3)(n^2-6n+3)\right\}^5$$

$$= \left\{x^5(n^2+3)(n^4+42n^2+9)(n^4+6n^3+18n^2+18n+9) + (n^2-3)\right\}^5$$

$$-\left\{x^5(n^2+3)(n^4+42n^2+9)(n^4-6n^3+18n^2-18n+9) + (n^2-3)\right\}^5$$

$$+\left\{6nx^5(n^2+4n-3)\right\}^5 - \left\{6nx^5(n^2-4n-3)\right\}^5$$

2/4 of algebraical addit...

If

(ii) $\dfrac{1+5x+9x^2}{1-8x-8x^2+x^3} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

or $\dfrac{\alpha_0}{x} + \dfrac{\alpha_1}{x^2} + \dfrac{\alpha_2}{x^3} + \ldots$

(iii) $\dfrac{2-26x-12x^2}{1-8x-8x^2+x^3} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots$

or $\dfrac{\beta_0}{x} + \dfrac{\beta_1}{x^2} + \dfrac{\beta_2}{x^3} + \ldots$

(iv) $\dfrac{2+8x-10x^2}{1-8x-8x^2+x^3} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

or $\dfrac{\gamma_0}{x} + \dfrac{\gamma_1}{x^2} + \dfrac{\gamma_2}{x^3} + \ldots$

then

$$\left.\begin{aligned} a_n^3 + b_n^3 &= c_n^3 + (-1)^n \\ \text{and} \quad \alpha_n^3 + \beta_n^3 &= \gamma_n^3 + (-1)^n \end{aligned}\right\}$$

Examples

$$135^3 + 138^3 = 179^3 - 1$$

$$11161^3 + 11468^3 = 14395^3 + 1$$

$$791^3 + 819^3 = 1010^3 - 1$$

$$65601^3 + 67402^3 = 83802^3 + 1$$

$$9^3 + 10^3 = 12^3 + 1$$

$$6^3 + 8^3 = 9^3 - 1$$

If $u_n = \dfrac{\sqrt{a^2+4}}{\left(\frac{a+\sqrt{a^2+4}}{2}\right)^n - \left(\frac{a-\sqrt{a^2+4}}{2}\right)^n}$

then $\dfrac{-c + b\cdot\frac{u_{n+1}}{u_n} + \sqrt{\left(c+b\cdot\frac{u_{n+1}}{u_n}\right)^2 + (-1)^n u_{n+1}^2}}{2}$

$$= \frac{1}{a+}\ \frac{1}{a+}\ \frac{1}{a+}\ \cdots\ \frac{1}{a+}\ \frac{b}{c+\frac{1}{a+\frac{b}{c+}}}$$

n of these

(83)

Page 378 $z^6 - 5z^3 + 3z^2 + 3z + 2$, Modular equation region.

$$z = \frac{f(\sqrt{-21})}{\sqrt{2}} + \frac{\sqrt{2}}{f(\sqrt{-21})}.$$

z ought to be $\frac{\left|f(\sqrt{-21})\right|^2}{\sqrt{2}} + \frac{\sqrt{2}}{\left|f(\sqrt{-21})\right|^2}$.

Page 500 $\left(x + \frac{1}{x}\right)^2 - \frac{5+\sqrt{41}}{2}\left(x + \frac{1}{x}\right) + \frac{7+\sqrt{41}}{2} = 0,$

$$x\sqrt{2} = f(\sqrt{-21})$$

$x\sqrt{2}$ ought to be $\left|f(\sqrt{-21})\right|^2$.

Webers Results

$f_1(\sqrt{-20}) = t^{5/8}$. $\quad t^8 - \frac{1+\sqrt{5}}{2}(2t^4+1) = 0$.

$f(\sqrt{-27}) = t^{3/2}$. $\quad t^9 - 3t^6 - 3t^3 - 1 = 0$

$f_1(\sqrt{-36}) = t^{3/2}$. $\quad t^6 - t^4 - \frac{3+\sqrt{13}}{2}(t^3+1) = 0$.

$f(\sqrt{-39}) = t^{3/2}$. $\quad 2t^{12} - 9t^9 - 8t^4 - 5 = \sqrt{29}(t^4+1)^2$.

$f(\sqrt{-35}) = t$ $\quad t^3 - 2 = (1+\sqrt{5})(t^2 - t)$.

$f_1(\sqrt{-36}) = t^{3\,8}/2$ $\quad t^6 - 4t^3 - 2 = 2\sqrt{3}(t^3+1)$.

$f_1(\sqrt{-38}) = t^{5/2}$ $\quad t^6 - 2t^4 - (2t^2+1)(1+\sqrt{2}) = 0$

$f(\sqrt{-37}) = t^3\sqrt{2}$ $\quad t^6 - \frac{3+\sqrt{13}}{2}(t^3+1) = 0$

$f(\sqrt{-41}) = t^{5/2}$ $\quad \left(t^3 + \frac{1}{t^3}\right)^2 - \frac{5+\sqrt{41}}{2}\left(t^2 + \frac{1}{t^2}\right) + \frac{7+\sqrt{41}}{2} = 0$

$f_1(\sqrt{-44}) = t^{5/8}$ $\quad t^{12} - (6+2\sqrt{11})t^8 + (8+2\sqrt{11})t^4 - (3+\sqrt{11}) = 0$

$f_1(\sqrt{-50}) = t^{5/2}$ $\quad t^3 - t^2 = \frac{1+\sqrt{5}}{2}(t+1)$

$f(\sqrt{-51}) = t^{3/2}$ $\quad t^9 - (4+\sqrt{17})t^6 - t^3 - 1 = 0$

$f_1(\sqrt{-52}) = t^{5/8}$ $\quad t^8 - 2(4+\sqrt{13})t^4 - \frac{3+\sqrt{13}}{2} = 0$

$f(\sqrt{-63}) = t\sqrt{2}$ $\quad \frac{t^6 + 3t^3 - 1}{t^6 - t^3 + 1} = \sqrt{\frac{7}{3}}$.

$f(\sqrt{-87}) = t^{3/2}$ $\quad t^9 - (13+2\sqrt{33})t^6 - (4+\sqrt{33})t^3 - 1 = 0$

l, m, n are ... integers ... of θ (real)

$$\theta = \frac{\sqrt{5}}{2}.$$

$$a = \frac{1}{\frac{\sqrt{5}}{2} - 1}, \quad b = \frac{\sqrt{5}}{\left(1 + \frac{\sqrt{5}}{2}\right)^{\frac{5}{2}}}$$

$$a^m b^n \theta = p_{m,n} + \varepsilon_{m,n}$$

where $-\frac{1}{2} < \varepsilon_{m,n} < \frac{1}{2}$ and $p_{m,n}$ ... integer. Then

$$\varepsilon_{m,n} = 0 \frac{5^{\frac{n}{2}}}{\left(\frac{5}{4} - 2\frac{\sqrt{5}}{2}\cos\frac{2\pi s}{5} + 1\right)^{\frac{m}{2}} \left(\frac{5}{16} + 2\frac{\sqrt{5}}{4}\cos\frac{4\pi s}{5} + 1\right)^{\frac{n}{4}}}$$

where s is the most unfavourable of the integers $1, 2, 3, 4$.

$$\theta = \sqrt{2}$$

$$a = \frac{1}{\sqrt{2} - 1}, \quad b = \frac{7}{(\sqrt{8} - 1)^7}, \quad c = \frac{\sqrt{2} + 1}{\sqrt{4} - \sqrt{2} + 1},$$

$$a^l b^m c^n \theta = p_{l,m,n} + \varepsilon_{l,m,n}$$

$$\varepsilon_{l,m,n} = 0 \frac{7^m \left(\sqrt{4} + 2\sqrt{2}\cos\frac{2\pi s}{5} + 1\right)^{2n}}{\left(\sqrt{4} - 2\sqrt{2}\cos\frac{2\pi s}{7} + 1\right)^{\frac{l}{2}} \left(\sqrt{64} - 2\sqrt{8}\cos\frac{6\pi s}{7} + 1\right)^{\frac{7m}{2}} \left(\sqrt{64} + 2\sqrt{8}\cos\frac{6\pi s}{7} + 1\right)^{\frac{n}{2}}},$$

where s is the most unfavourable of the integers $1, 2, 3, 4, 5, 6$

Add 1.14.a 9+ 18 (85) (85)

$4 = 5^-.$ $\quad \dfrac{\sqrt[5]{3+2y} - \sqrt[5]{4-4y}}{\sqrt[5]{3+2y} + \sqrt[5]{4-4y}} = 2 + y + y^2 + y^3$

$y^5 = 2.$ $\quad \dfrac{\sqrt{y+3} + \sqrt{5y-5}}{\sqrt{y+3} - \sqrt{5y-5}} = y + y^2$

$y^5 = 2.$ $\quad \dfrac{\sqrt{y^2+1} + \sqrt{4y-3}}{\sqrt{y^2+1} - \sqrt{4y-3}} = \frac{1}{5}(1 + y^2 + y^3 + 11y^2)^2$

$y^5 = 3.$ $\quad \dfrac{\sqrt{y^2+1} + \sqrt{5y-5}}{\sqrt{y^2+1} - \sqrt{5y-5}} = \frac{1}{y} + y + y^2 + y^3$

$y^5 = 2.$ $\quad \sqrt{1+y^2} = \dfrac{y^4 + y^3 + y - 1}{\sqrt{5}}$ $\quad \times$

$y^5 = 2.$ $\quad \sqrt{4y-3} = \dfrac{11y^8 + 11y^7 - 11y^6 - 1}{\sqrt{5}}$ $\quad \times$

$y^5 = 3.$ $\quad \sqrt[3]{2-y^3} = \dfrac{1+y-y^2}{\sqrt[3]{5}}.$ $\quad y = 3.$

$y^5 = 2.$ $\quad \sqrt[5]{1+y+y^3} = \dfrac{\sqrt{1+y^2}}{\sqrt[10]{5}}$ $\quad y^5 + 5y^3 + 5y + 2 = 0.$

$$\sqrt[3]{\tfrac{1}{3}} + \sqrt[3]{\tfrac{5}{3}} = \sqrt{\frac{\sqrt[3]{5}-1}{2-\sqrt[3]{5}}} = \sqrt[3]{\frac{3+\sqrt[3]{5}}{\sqrt[3]{5}-1}} = \sqrt[5]{\frac{3\sqrt[3]{3}+\sqrt[3]{15}}{2-\sqrt[3]{5}}} \;.$$

$$\left\{\sqrt[3]{(a+b)(a^2+b^2)} - a\right\}\left\{\sqrt[3]{(a+b)(a+b^2)} - b\right\} = \frac{\sqrt[3]{(a+b)^2} - \sqrt[3]{a^2+b^2}}{\sqrt[3]{(a+b)^2} + \sqrt[3]{a^2+b^2}}(a^2+ab+b^2).$$

$$\frac{\left(\sqrt{a^2+ab+b^2} - a\right)\left(\sqrt{a^2+ab+b^2} - b\right)}{a+b-\sqrt{a^2+ab+b^2}} = a+b.$$

$$\left\{c + \sqrt{(b+c)(c+a)}\right\}\left\{a + \sqrt{(c+a)(a+b)}\right\}\left\{-b + \sqrt{(a+b)(b+c)}\right\}$$

$$2(ab+bc+ca)\left(\frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{\text{etc.}}\right)^2.$$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^4})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^2})(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^4})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^4})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^4})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^4})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2^2})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^2})$

$2^{31}(1+\frac{1}{2^4})(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^4})(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2^4})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^4})$

$2^{31}(1+\frac{1}{2^8})(1+\frac{1}{2^{16}})$

$2^{31}(1+\frac{1}{2^8})$

$2^{31}(1+\frac{1}{2^{16}})$

2^{31}

$$\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+4\left\{1\cdot\frac{v\sin\theta}{1-v}-\frac{1}{3}\cdot\frac{v^3\sin 3\theta}{1-v^3}+\frac{1}{5}\cdot\frac{v^5\sin 5\theta}{1-v^5}-\cdots\right\}$$

$$=\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$

where
$$\frac{2K\theta}{\pi}=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

$$\theta'+2\left\{1\cdot\frac{v\sinh 2\theta'}{1+v^2}+\frac{1}{2}\cdot\frac{v^3\sinh 6\theta'}{1+v^6}+\frac{1}{3}\cdot\frac{v^5\sinh 10\theta'}{1+v^{10}}+\cdots\right\}$$

$$=\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right),$$

where
$$\frac{2K\theta'}{\pi}=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}$$

If $2u = 11 + \frac{1}{x} \cdot \left\{ \frac{(1-x)(1-x^4)(1-x^5)\ldots}{(1-x^2)(1-x^3)(1-x^6)\ldots} \right\}^6$

and $2v = 1 + \frac{1}{x^2} \left\{ \frac{(1-x^4)(1-x^8)(1-x^{12})\ldots}{(1-x^1)(1-x^5)(1-x^9)\ldots} \right\}$.

then $\sqrt[5]{\sqrt{u^2+1} - u} = \sqrt{v^2+1} - v$

$= x^{\frac{1}{5}} \cdot \frac{(1-x)(1-x^6)(1-x^{11})\ldots}{(1-x^2)(1-x^7)(1-x^{12})\ldots} \cdot \frac{(1-x^4)(1-x^8)(1-x^{14})\ldots}{(1-x^3)(1-x^8)(1-x^{13})\ldots}$

$1 + \frac{x}{1-x} + \frac{x^5}{(1-x)(1-x^2)} +$

$= \frac{1}{(1-x)(1-x^6)(1-x^{11})\ldots} \cdot \frac{1}{(1-x^5)(1-x^9)(1-x^{14})\ldots}$

$1 + \frac{x^2}{1-x^2} + \frac{x^6}{(1-x^2)(1-x^3)} + \frac{x^{12}}{(1-x^2)(1-x^3)(1-x^4)} + \ldots$

$= \frac{1}{(1-x^4)(1-x^7)\ldots} \cdot \frac{1}{(1-x^3)(1-x^8)\ldots}$

$$f(x) = \sqrt[5]{x}\,\frac{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})\cdots\cdots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})\cdots\cdots}$$

$$\left(= \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\cfrac{x^4}{1+\cdots\cdots}}}}} \right)$$

I If $P = f(x)$ and $Q = f(-x)$, then

$$PQ(P+Q)^4 + P^2Q^2(P+Q)^2 + 2P^3Q^3 = (P+Q)(1-P^5Q^5)$$

II If $P = f(x)$ and $Q = f(x^2)$, then

$$P^2 - Q + PQ^2(P^2+Q) = 0$$

III If $P = f(x)$ and $Q = f(x^3)$, then

$$(P^3 - Q)(1+PQ^3) + 3P^2Q^2 = 0$$

IV If $P = f(x)$ and $Q = f(x^4)$, then

$$(P^5+Q^5)(PQ-1) + PQ + P^5Q^5 = 5P^2Q^2(PQ-1)^2.$$

V If $P = f(x)$ and $Q = f(x^5)$, then

$$P^5 = Q\cdot\frac{1-2Q+4Q^2-3Q^3+Q^4}{1+3Q+4Q^2+2Q^3+Q^4}.$$

$$n \equiv 15 \pmod{16}$$

15, 31, 47, 79, 95, 143, 191.

$$n \equiv 7 \pmod{16}$$

$$P = 1 - (h\ell)^{\frac{1}{2}} - (h'\ell')^{\frac{1}{2}}.$$

$$Q = \sqrt{\left(\frac{1 + h\ell + h'\ell'}{2}\right)} - (h\ell)^{\frac{1}{2}} - (h'\ell')^{\frac{1}{2}} + (hh'\ell\ell')^{\frac{1}{2}}.$$

$$R = 4(hh'\ell\ell')^{\frac{1}{4}}$$

7. $P^2 = Q = 0$

23. $P^2 = Q = R^{\frac{2}{3}}$.

39. $Q(P^2 - Q) = PR$.

55. $Q(P^2 - Q)^2 = R(P^2 - R)$.

71. $P^2 - Q = PR^{\frac{1}{3}}$.

103.

119. $(P^2 - Q)^2 = QR^{\frac{1}{3}}(P - R^{\frac{1}{3}})$.

167.

1. $n = 2$. $(1 - k')(1 - \ell) = 2k'\ell$.

$n = 4$ $(1 - \sqrt{k'})(1 - \sqrt{\ell}) = 2\sqrt{(k'\ell)}$.

$n = 8$ $(1 - \sqrt{k'})(1 - \sqrt{\ell}) = 2(k'\ell)^{\frac{1}{4}}$

2. $n = 2$ $\frac{1}{2}M^2 = \frac{1+\ell}{1+k'} = \frac{1+\ell^2}{1+k'^2}$.

$n = 4$ $\frac{1}{2}M = \frac{1+\sqrt{\ell}}{1+\sqrt{k'}} = \frac{1+\ell}{1+k'}$.

$n = 16$ $\frac{1}{2}\sqrt{M} = \frac{1+\sqrt{\ell}}{1+\sqrt{k'}}$.

3. $n = 2$ $\begin{cases} Mk' + \ell = 1. & M^2k' + \ell^2 = 1. \\ \frac{2}{M}\ell + k' = 1. & \frac{4}{M^2}\ell + k'^2 = 1. \end{cases}$

$n = 4$ $\begin{cases} \sqrt{M}\sqrt{k'} + \sqrt{\ell} = 1. & M\sqrt{k'} + \ell = 1. \\ \frac{2}{\sqrt{M}}\sqrt{\ell} + \sqrt{k'} = 1. & \frac{4}{M}\sqrt{\ell} + k' = 1 \end{cases}$

$n = 8$ $\begin{cases} \sqrt{M}(k')^{\frac{1}{4}} + \sqrt{\ell} = 1. \\ 2\sqrt{\left(\frac{2}{M}\right)}\ell^{\frac{1}{4}} + \sqrt{k'} = 1. \end{cases}$

4. $n = 3$. $\sqrt{(k\ell)} + \sqrt{(k'\ell')} = 1$

$n = 7$. $(k\ell)^{\frac{1}{2}} + (k'\ell')^{\frac{1}{2}} = 1$

$n = 15$. $(k\ell)^{\frac{1}{8}}\left[\left\{(1+k)(1+\ell)\right\}^{\frac{1}{4}} + \left\{(1-k)(1-\ell)\right\}^{\frac{1}{4}}\right]$

$+ (k'\ell')^{\frac{1}{8}}\left[\left\{(1+k')(1+\ell')\right\}^{\frac{1}{4}} + \left\{(1-k')(1-\ell')\right\}^{\frac{1}{4}}\right] = \sqrt{2}$.

$n = 31$. $(k\ell)^{\frac{1}{16}}\left[\left\{(1+k)(1+\ell)\right\}^{\frac{1}{8}}\sqrt{\left\{1 + \sqrt{(k\ell)} + \sqrt{\{(1-k)(1-\ell)\}}\right\}}\right.$

$\left. + \left\{(1-k)(1-\ell)\right\}^{\frac{1}{8}}\sqrt{\left\{1 + \sqrt{(k\ell)} + \sqrt{\{(1+k)(1+\ell)\}}\right\}}\right]$

$+ (k'\ell')^{\frac{1}{16}}\left[\left\{(1+k')(1+\ell')\right\}^{\frac{1}{8}}\sqrt{\left\{1 + \sqrt{(k'\ell')} + \sqrt{\{(1-k')(1-\ell')\}}\right\}}\right.$

$\left. + \left\{(1-k')(1-\ell')\right\}^{\frac{1}{8}}\sqrt{\left\{1 + \sqrt{(k'\ell')} + \sqrt{\{(1+k')(1+\ell')\}}\right\}}\right]$

$= 2^{\frac{3}{4}}$.

5. $n = 7$ or $\frac{5}{3}$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = (he)^{\frac{1}{2}} + (h'e')^{\frac{1}{2}} - (he\,h'e')^{\frac{1}{2}}$.

$n = 15$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = (he)^{\frac{1}{2}} + (h'e')^{\frac{1}{2}} + (he\,h'e')^{\frac{1}{2}}$.

$n = 23$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = \frac{1}{2}\left\{1 + \sqrt{(he)} + \sqrt{(h'e')}\right\} + (he\,h'e')^{\frac{1}{3}}$.

$n = 31$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = 1 + \sqrt{(he)} + \sqrt{(h'e')} - (he)^{\frac{1}{2}} - (h'e')^{\frac{1}{2}} - (he\,h'e')^{\frac{1}{2}}$.

$n = 47$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = \frac{1}{2}\left\{1 + \sqrt{(he)} + \sqrt{(h'e')}\right\} + \left(\frac{1}{16}\,he\,h'e'\right)^{\frac{1}{12}}\left\{1 + (he)^{\frac{1}{2}} + (h'e')^{\frac{1}{2}}\right\}$.

$n = 71$. $\sqrt{\left(\frac{1+he+h'e'}{2}\right)} = 1 + \sqrt{(he)} + \sqrt{(h'e')} - (he)^{\frac{1}{2}} - (h'e')^{\frac{1}{2}} + (he\,h'e')^{\frac{1}{2}} + (256\,he\,h'e')^{\frac{1}{12}}\left\{(he)^{\frac{1}{2}} + (h'e')^{\frac{1}{2}} - 1\right\}$.

6. $n = 3$
$$\begin{cases} M^2 = \frac{e}{h} + \frac{e'}{h'} - \frac{ee'}{hh'}. \\ \frac{9}{M^2} = \frac{h}{e} + \frac{h'}{e'} - \frac{hh'}{ee'}. \end{cases}$$

$n = 5$.
$$\begin{cases} M = \sqrt{\left(\frac{e}{h}\right)} + \sqrt{\left(\frac{e'}{h'}\right)} - \sqrt{\left(\frac{ee'}{hh'}\right)}. \\ \frac{5}{M} = \sqrt{\left(\frac{h}{e}\right)} + \sqrt{\left(\frac{h'}{e'}\right)} - \sqrt{\left(\frac{hh'}{ee'}\right)} \end{cases}$$

$n = 7$.
$$\begin{cases} M^2 = \frac{e}{h} + \frac{e'}{h'} - \frac{ee'}{hh'} - 8\left(\frac{ee'}{hh'}\right)^{\frac{3}{3}} \\ \frac{49}{M^2} = \frac{h}{e} + \frac{h'}{e'} - \frac{hh'}{ee'} - 8\left(\frac{hh'}{ee'}\right)^{\frac{3}{3}} \end{cases}$$

$n = 9$.
$$\begin{cases} \sqrt{M} = \left(\frac{e}{h}\right)^{\frac{1}{2}} + \left(\frac{e'}{h'}\right)^{\frac{1}{2}} - \left(\frac{ee'}{hh'}\right)^{\frac{1}{2}} \\ \frac{3}{\sqrt{M}} = \left(\frac{h}{e}\right)^{\frac{1}{2}} + \left(\frac{h'}{e'}\right)^{\frac{1}{2}} - \left(\frac{hh'}{ee'}\right)^{\frac{1}{2}} \end{cases}$$

$$m = 13 \qquad \begin{cases} M = \sqrt{\left(\frac{c}{h}\right)} + \sqrt{\left(\frac{c'}{h'}\right)} - \sqrt{\left(\frac{c\,c'}{h\,h'}\right)} - 4\left(\frac{c\,c'}{h\,h'}\right)^{\frac{1}{3}} \\ \frac{13}{M} = \sqrt{\left(\frac{h}{c}\right)} + \sqrt{\left(\frac{h'}{c'}\right)} - \sqrt{\left(\frac{h\,h'}{c\,c'}\right)} - 4\left(\frac{h\,h'}{c\,c'}\right)^{\frac{1}{3}} \end{cases}$$

$$m = 17 \qquad \begin{cases} M = \sqrt{\left(\frac{c}{h}\right)} + \sqrt{\left(\frac{c'}{h'}\right)} + \sqrt{\left(\frac{c\,c'}{h\,h'}\right)} - 2\left(\frac{c\,c'}{h\,h'}\right)^{\frac{1}{2}}\left\{1 + \left(\frac{c}{h}\right)^{\frac{1}{2}} + \left(\frac{c'}{h'}\right)^{\frac{1}{2}}\right\} \\ \frac{17}{M} = \sqrt{\frac{h}{c}} + \sqrt{\left(\frac{h'}{c'}\right)} + \sqrt{\left(\frac{h\,h'}{c\,c'}\right)} - 2\left(\frac{h\,h'}{c\,c'}\right)^{\frac{1}{2}}\left\{1 + \left(\frac{h}{c}\right)^{\frac{1}{2}} + \left(\frac{h'}{c'}\right)^{\frac{1}{2}}\right\} \end{cases}$$

$$m = 25 \qquad \begin{cases} \sqrt{M} = \left(\frac{c}{h}\right)^{\frac{1}{2}} + \left(\frac{c'}{h'}\right)^{\frac{1}{2}} - \left(\frac{c\,c'}{h\,h'}\right)^{\frac{1}{2}} - 2\left(\frac{c\,c'}{h\,h'}\right)^{\frac{1}{6}} \\ \frac{5}{\sqrt{M}} = \left(\frac{h}{c}\right)^{\frac{1}{2}} + \left(\frac{h'}{c'}\right)^{\frac{1}{2}} - \left(\frac{h\,h'}{c\,c'}\right)^{\frac{1}{2}} - 2\left(\frac{h\,h'}{c\,c'}\right)^{\frac{1}{6}} \end{cases}$$

(2)

(1.13) $(1+2x+2x^4+2x^9+\dots)^6$

$= 1 - 4\left(\frac{1^2x}{1-x} - \frac{2^2x^2}{1-x^2} + \frac{3^2x^3}{1-x^3} - \dots\right)$

$+ 16\left(\frac{1^2x}{1+x^2} + \frac{2^2x^2}{1+x^4} + \frac{3^2x^3}{1+x^6} + \dots\right).$

(1.14) $(1+2x+2x^4+2x^9+\dots)^8$

$= 1 + 16\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \dots\right)$

(1.21) $\dfrac{(1+2x+2x^4+2x^9+\dots)^3}{1+2x^3+2x^{12}+2x^{27}+\dots}$

$= 1 + 6\left(\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^4}{1+x^4} - \frac{x^5}{1+x^5} + \dots\right)$

(1.22) $\dfrac{(1+2x^3+2x^{12}+2x^{27}+\dots)^3}{1+2x+2x^4+2x^9+\dots}$

$= 1 - 2\left(\frac{3x}{1+x} - \frac{x^2}{1-x^2} + \frac{x^4}{1+x^4} - \frac{x^5}{1-x^5} + \dots\right)$

(1.31) $\dfrac{(1+x+x^2+x^5+x^{10}+\dots)^3}{1+x^3+x^9+x^{18}+x^{30}+\dots}$

$= 1 + 3\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \dots\right)$

(1.32) $x\dfrac{(1+x^5+x^9+x^{18}+x^{30}+\dots)^3}{1+x+x^2+x^6+x^{10}+\dots}$

$= \frac{x}{1-x^2} - \frac{x^3}{1-x^4} + \frac{x^5}{1-x^6} - \frac{x^{15}}{1-x^{10}} + \dots$

(1.41) $\dfrac{(1-x)^3(1-x^2)^3(1-x^3)^3(1-x^4)^3\dots}{(1-x^3)(1-x^6)(1-x^9)(1-x^{12})\dots}$

$= 1 - 3\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^5}{1-x^5} - \frac{x^{10}}{1-x^{10}} + \dots\right) +$

(9)

$$9\left(\frac{x^3}{1-x^3}-\frac{x^6}{1-x^6}+\frac{x^{12}}{1-x^{12}}-\frac{x^{15}}{1-x^{15}}+\cdots\right).$$

(1.43) $$\frac{x(1-x^9)^3(1-x^{18})^3(1-x^{27})^3\cdots}{(1-x^3)(1-x^6)(1-x^9)\cdots}$$

$$=\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^5}{1-x^5}+\cdots$$

$$-\left(\frac{x^3}{1-x^3}-\frac{x^6}{1-x^6}+\frac{x^{12}}{1-x^{12}}-\frac{x^{15}}{1-x^{15}}+\cdots\right).$$

(1.51) $$\frac{(1-x)^5(1+x^2)^5(1-x^3)^5\cdots}{(1-x^{15})(1-x^{10})(1-x^{15})\cdots}$$

$$=1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}-\frac{4x^4}{1-x^4}\right.$$

$$\left.+\frac{6x^6}{1-x^6}-\frac{7x^7}{1-x^7}-\frac{8x^8}{1-x^8}+\cdots\right).$$

(1.52) $$\frac{x(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}$$

$$=\frac{x^2}{(1-x^2)^2}-\frac{x^3}{(1-x^3)^2}-\frac{x^5}{(1-x^5)^2}+\frac{x^7}{(1-x^7)^2}$$

$$+\frac{x^{12}}{(1-x^{12})^2}-\frac{x^{13}}{(1-x^{13})^2}-\frac{x^{15}}{(1-x^{15})^2}+\cdots\cdots$$

(1.61) $$(1+2x+2x^4+\cdots)(1+2x^3+2x^{12}+\cdots)$$

$$=1+2\left(\frac{x}{1-x}-\frac{x^2}{1+x^2}+\frac{x^4}{1+x^4}-\frac{x^5}{1-x^5}+\cdots\right)$$

(1.62) $$(1+2x+2x^4+\cdots)^2(1+2x^3+2x^{12}+\cdots)^2$$

$$=1+4\left(\frac{x}{1+x}+\frac{2x^2}{1-x^2}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1+x^5}-\cdots\right)$$

(1.71) $$(1+2x+2x^4+\cdots)(1+2x^7+2x^{28}+\cdots)$$

$$=1+2\left(\frac{x}{1+x}+\frac{x^2}{1-x^2}-\frac{x^3}{1+x^3}+\frac{x^4}{1-x^4}-\frac{x^5}{1+x^5}-\frac{x^6}{1-x^6}\right.$$

$$\left.+\frac{x^7}{1+x^7}+\frac{x^8}{1-x^8}-\frac{x^{10}}{1-x^{10}}+\cdots\right).$$

(1.79) $x(1+x+x^3+x^6+\dots)(1+x^7+x^{14}+x^{21}+\dots)$

$$= \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^{11}}{1-x^{11}} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}}$$

$$- \frac{x^{13}}{1-x^{13}} + \frac{x^{17}}{1-x^{17}} - \frac{x^{17}}{1-x^{17}} - \dots$$

(1.81) $$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+n^2} = 1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^5}{1-x^5}\right.$$

(1.82) $$\left\{\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+n^2}\right\}^2 = 1 + 12\left(\frac{x}{1-x} + \frac{4x^2}{1-x^2} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \dots\right)$$

(1.83) $$\left\{\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+n^2}\right\}^3 = 1 - 9\left(\frac{1^2x}{1-x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \dots\right)$$

$$+ 27\left(\frac{1^2x}{1+x+x^2} + \frac{2^2x^2}{1+x^2+x^4} + \frac{3^2x^3}{1+x^3+x^6} + \dots\right)$$

(1.84) $$\left\{\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+n^2}\right\}^4 = 1 + 24\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \dots\right)$$

$$+ 8\left(\frac{3^3x^3}{1-x^3} + \frac{6^3x^6}{1-x^6} + \frac{9^3x^9}{1-x^9} + \dots\right).$$

(1.91) $$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+2n^2} = 1 + 2\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5}\right.$$

$$\left. - \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} - \frac{x^{10}}{1-x^{10}} + \dots\right)$$

(1.92) $$\left\{\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{m^2+mn+2n^2}\right\}^2 = 1 + 4\left(\frac{x}{1-x} + \frac{3x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6}\right.$$

$$\left. + \frac{8x^8}{1-x^8} + \dots\right)$$

(3.12) $\psi^4(x) = 1 + 8\left(\frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \cdots\right)$

(3.13) $\psi^6(x) = 1 - 4\left(\frac{1^2x}{1-x} - \frac{3^2x^3}{1-x^3} + \frac{5^2x^5}{1-x^5} - \cdots\right)$
$+ 16\left(\frac{1^2x}{1+x^2} + \frac{2^2x^2}{1+x^4} + \frac{3^2x^3}{1+x^6} + \cdots\right)$

(3.14) $\psi^8(x) = 1 + 16\left(\frac{1^3x}{1+x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1+x^3} + \cdots\right)$

(3.21) $x^{\frac{1}{4}}\psi^2(x) = \frac{x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}} + \frac{x^{\frac{5}{2}}}{1-x^{\frac{5}{2}}} - \cdots$

(3.22) $x^{\frac{1}{2}}\psi^4(x) = \frac{x^{\frac{1}{2}}}{1-x} + \frac{3x^{\frac{3}{2}}}{1-x^3} + \frac{5x^{\frac{5}{2}}}{1-x^5} + - \cdots$

(3.23) $x^{\frac{3}{4}}\psi^6(x) = \frac{1}{16}\left(\frac{1^2x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}} + \frac{3^2x^{\frac{3}{2}}}{1+x^{\frac{3}{2}}} + \frac{5^2x^{\frac{5}{2}}}{1+x^{\frac{5}{2}}} + \cdots\right)$
$- \frac{1}{16}\left(\frac{1^2x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}} - \frac{3^2x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}} + \frac{5^2x^{\frac{5}{2}}}{1-x^{\frac{5}{2}}} - \cdots\right)$

(3.24) $x\psi^8(x) = \frac{1^3x}{1-x^2} + \frac{2^3x^2}{1-x^4} + \frac{3^3x^3}{1-x^6} + - \cdots$

These are of course well known formulae for the number of representations of a number as the sum of 2, 4, 6 and 8 squares or triangular numbers. There are also various other arithmetical problems in which the partition method gives the actual value. I shall quote a few examples and reserve the discussion of these to another paper.

(3.31) $S = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} x^{\mu^2+\mu\nu+\nu^2} = 1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4}\right.$
$\left. - \frac{x^5}{1-x^5} + \cdots\right)$

(3.32) $S^2 = 1 + 12\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{4x^4}{1-x^4} + \frac{5-x^5}{1-x^5} + \cdots\right)$

(3.33) $S^3 = 1 - 9\left(\frac{1^2x}{1-x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1-x^5} + \cdots\right)$
$+ 27\left(\frac{1^2x}{1+x+x^2} + \frac{2^2x^2}{1+x^2+x^4} + \frac{3^2x^3}{1+x^3+x^6} + \cdots\right)$

(3.34) $S^4 = 1 + 24\left(\frac{1^3x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \cdots\right)$
$+ 8\left(\frac{3^3x^3}{1-x^3} + \frac{6^3x^6}{1-x^6} + \frac{9^3x^9}{1-x^9} + \cdots\right)$

(7)

$$(3\cdot41)\quad T=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}x^{\mu^2+\mu\nu+2\nu^2}=1+2\left(\frac{x}{1-x}+\frac{x^2}{1-x^2}-\frac{x^3}{1-x^3}+\frac{x^4}{1-x^4}-\frac{x^{10}}{1-x^{10}}-\frac{x^6}{1-x^6}+\frac{x^8}{1-x^8}+\cdots\right)$$

$$(3\cdot42)\quad T^2=1+4\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{5x^{10}}{1-x^{10}}+\frac{6x^6}{1-x^6}+\frac{8x^8}{1-x^8}+\cdots\right),$$

where 1, 2, 3, ... are the natural numbers without the multiples of 7.

The following formulas seem to be of some interest in itself though they haven't got a simple arithmetical interpretation as the preceding ones.

$$(3\cdot51)\quad \frac{\phi^3(x)}{\phi(x^3)}=1+6\left(\frac{x}{1-x}+\frac{x^2}{1+x^2}-\frac{x^4}{1+x^4}-\frac{x^5}{1-x^5}+\cdots\right)$$

$$(3\cdot52)\quad \frac{\psi^3(x^3)}{\phi(x)}=1-2\left(\frac{x}{1+x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^{10}}{1+x^{10}}-\cdots\right)$$

$$(3\cdot61)\quad \frac{\psi^3(x)}{\psi(x^3)}=1+3\left(\frac{x}{1-x}-\frac{x^{10}}{1-x^{10}}+\frac{x^7}{1-x^7}-\cdots\right)$$

$$(3\cdot62)\quad \frac{x\psi^3(x^3)}{\psi(x)}=\frac{x}{1-x^2}-\frac{x^2}{1-x^4}+\frac{x^4}{1-x^6}-\frac{x^{10}}{1-x^{10}}+\cdots$$

$$(3\cdot71)\quad \frac{f^3(x)}{f(x^3)}=1-3\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^{10}}{1-x^{10}}+\cdots\right)+9\left(\frac{x^3}{1-x^3}-\frac{x^6}{1-x^6}+\frac{x^{12}}{1-x^{12}}-\frac{x^{15}}{1-x^{15}}+\cdots\right)$$

$$(3\cdot72)\quad \frac{xf^3(x^3)}{f(x^3)}=\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\frac{x^{10}}{1-x^{10}}+\cdots-\left(\frac{x^3}{1-x^3}-\frac{x^6}{1-x^6}+\frac{x^{12}}{1-x^{12}}-\frac{x^{15}}{1-x^{15}}+\cdots\right)$$

$$(3\cdot81)\quad \frac{f^5(x)}{f(x^5)}=1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}-\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{6x^6}{1-x^6}-\frac{7x^7}{1-x^7}-\cdots\right)$$

$$(3\cdot82)\quad \frac{xf^5(x^5)}{f(x)}=\frac{x}{(1-x)^2}-\frac{x^2}{(1-x^2)^2}-\frac{x^3}{(1-x^3)^2}+\frac{x^4}{(1-x^4)^2}+\frac{x^6}{(1-x^6)^2}-\frac{x^7}{(1-x^7)^2}-\cdots\cdots$$

2. The reasons for taking the series (12)

$$1 + \frac{x}{1-x} + \frac{x^6}{(1-x)(1-x^5)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \cdots$$

$$= \frac{1}{(1-x)(1-x^6)(1-x^{11})\cdots} \cdot \frac{1}{(1-x^4)(1-x^9)\cdots(1-x^{14})\cdots}$$

are the following.

(a) Mr. Macmahon has verified up to x^{55} and found the result correct up to that term.

(b) It can be shewn independently that,

$$\log\{\text{L.H.S of (12)}\} \sim \frac{\pi^2}{15(1-x)} \quad \text{as } x \to 1$$

as well as

$$\log\{\text{R.H.S of (12)}\} \sim \frac{\pi^2}{15(1-x)} \quad \text{as } x \to 1.$$

(c) The numerical results of the cont. fraction go to prove the truth of the result.

(d). If $v = \frac{x}{1+} \frac{x^5}{1+} \frac{x^{10}}{1+} \frac{x^{15}}{1+} \cdots$

then $\frac{1}{v} - v - 1$ vanishes when x is of the form $e^{i\pi \cdot \frac{m}{n}}$ when m and n are any two integers prime to each other, except when n is a multiple of 25.

As a matter of fact, if v is assumed product of the c.f. $\frac{x}{1+} \frac{x^5}{1+} \frac{x^{10}}{1+} \cdots$

$$1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^8}{(1-x^4)(1-x^8)(1-x^{12})} + \cdots$$

$$(1-x^2)(1-x^6)(1-x^{10})(1-x^{14}) \cdots$$

$$(1-x)(1-x^6)(1-x^{11}) \cdots \times (1-x^4)(1-x^8)(1-x^{12}) \cdots$$

$$x + \frac{x^4}{1-x^5} + \frac{x^9}{(1-x^5)(1-x^{10})} + \frac{x^{16}}{\cdots}$$

$$(1-x^2)(1-x^6)(1-x^{10})(1-x^{14}) \cdots \times (1-x^3)(1-x^8)(1-x^{13}) \cdots$$

$$1 + \frac{a x^{m+n}}{1-x} + \frac{a^2 x^{2m+4n}}{(1-x)(1-x^2)} + \frac{a^3 x^{3m+9n}}{(1-x)(1-x^2)(1-x^3)}$$

$$\sim \frac{z^m e^{-\frac{1}{\log x}\int_0^{\log \frac{1}{z}} \frac{-a}{e^a}\, da}}{\sqrt{z + zm(1-z)}} \qquad \text{as } x \to 0$$

where $a z^{2m} + z = 1$.

$$1 + \frac{x}{1-x} = \frac{1}{1-x}$$

$$1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} = \frac{1-x^2+x^4}{(1-x)(1-x^2)}$$

$$+ \frac{x^9}{(1-x)(1-x^2)(1-x^3)} \quad \left| \quad \frac{1-x^2-x^3+x^4+x^5-x^7+x^9}{(1-x)(1-x^2)(1-x^3)} \right.$$

$$+ \frac{x^{16}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \quad \left| \quad \frac{1-x^2-x^3+x^6+x^6-x^8+x^{11}-x^{13}+x^{16}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \right.$$

$$+ \frac{x^{25}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} \quad \left| \quad \frac{1-x^2-x^3+x^6+x^7-x^{10}+x^{18}-x^{21}+x^{25}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} \right.$$

$$+ \frac{x^{36}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)} \quad \left| \quad \frac{1-x^2-x^3+x^7+x^8+x^9-x^{10}-x^{12}-x^{13}+x^{16}+x^{18}-x^{21}-x^{24}+x^{25}+x^{27}-x^{31}+x^{36}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)} \right.$$

Hence $(1-x)(1-x^2)(1-x^3)\cdots\cdots \left\{ 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \cdots \right\}$

$= (1-x^7)(1-x^8)(1-x^9)\cdots\cdots \left\{ 1-x^2-x^3+x^7+x^8+x^9-x^{10}-x^{12}-x^{13}+x^{16} \right.$
$\left. +x^{18}-x^{21}-x^{24}+x^{25}+x^{27}-x^{31}+x^{36} \right\}$

$= (1-x^8)(1-x^9)\cdots\cdots \left\{ 1-x^2-x^3+x^8+x^9-x^{12}-x^{13}-x^{14}-x^{15}+x^{17}+x^{18} \right.$
$\left. +x^{19}+x^{20}-x^{21}-x^{23}-x^{24}+x^{27}+x^{28}-x^{32}-x^{34}+x^{36} \right\}$

$= 1 - \frac{x^8}{1-x} + \frac{x^{17}}{(1-x)(1-x^2)} - \frac{x^{27}}{(1-x)(1-x^2)(1-x^3)} + \frac{x^{38}}{(1-x)} \cdots$

$= 1 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} - x^{16} + x^{17} + x^{18} + 2x^{19} + 2x^{20} + 2x^{21} + 2x^{22}$
$+ 3x^{23} + 3x^{24} + 4x^{25} + 4x^{26} + 5x^{27} + 5x^{28} + 6x^{29} + 6x^{30} + 7x^{31} + 7x^{32}$
$+ 8x^{33} + 8x^{34} + 9x^{35} + 9x^{36} + \cdots$
$- x^{37} - x^{38} - 2x^{39} - 3x^{40} - 4x^{41} - 5x^{42} - 7x^{43} - 8x^{44} - 10x^{45} - 12x^{46}$

Hence $(1-x)(1-x^2)\cdots\left\{1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\cdots\right\}$

coeff. of x^{36} in

$$\left(1-x^2-x^3+x^8+2x^9-x^{12}-x^{13}-x^{15}+x^{17}+x^{18}+x^{19}+x^{20}-x^{21}\right.$$
$$\left.-x^{23}-x^{24}+x^{27}+x^{28}-x^{32}-x^{35}+x^{36}\right)$$
$$\times\left(1+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}+2x^{21}+2x^{22}\right.$$
$$\left.+3x^{23}+3x^{24}+4x^{25}+4x^{26}+4x^{27}+4x^{28}+4x^{29}+3x^{30}+3x^{31}\right.$$
$$\left.+2x^{32}+x^{33}+x^{34}+x^{36}\right)$$
$$=1-x^2-x^8+x^9+x^{11}-x^{21}-x^{24}$$ (11)

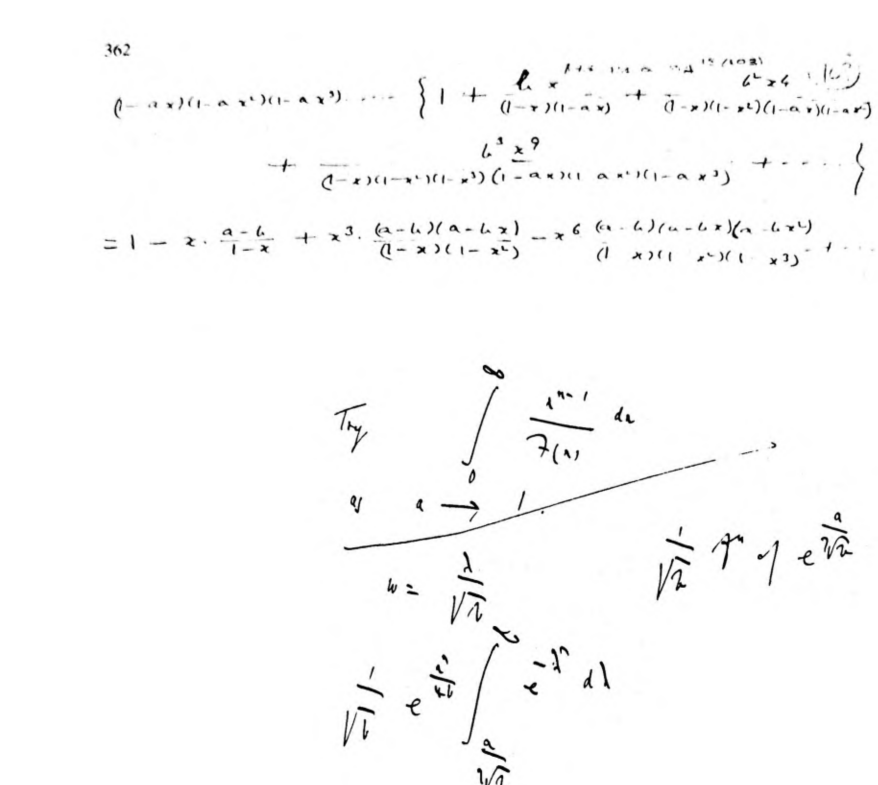
$$=(1-x^2)(1-x^7)(1-x^{12})(1-x^{17})(1-x^{28})(1-x^{32})(1-x^{36})\cdots$$
$$\times(1-x^3)(1-x^8)(1-x^{13})(1-x^{18})(1-x^{23})(1-x^{28})(1-x^{33})$$
$$\times(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})(1-x^{25})(1-x^{30})(1-x^{35})\cdots$$

Therefore $1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\cdots$

$$=\frac{1}{(1-x^{16})(1-x^{11})(1-x^9)(1-x)}\cdots$$
$$\times(1-x^{17})(1-x^{14})(1-x^8)(1-x^5)\cdots$$

(11) proof. let $p=x^{\frac{1}{2}}$ and $q=x^{\frac{1}{2}}$, then

$$1-x^2-x^8+x^9+x^{11}-x^{21}-x^{24}+\cdots$$
$$=1-p^4(q+q^3)+p^8(q^2+q^6)-p^9(q^3+q^9)+\cdots$$
$$=(1-pq)(1-p^3q)\cdots$$
$$(1-\tfrac{p}{q})(1-\tfrac{p^3}{q})\cdots$$
$$(1-p^2)(1-p^4)(1-p^6)\cdots$$



Additional results

Add to a §4[18 (104)] (104)

$$\frac{x + \frac{x^4}{1-x^4} + \frac{x^9}{(1-x^4)(1-x^8)} + \frac{x^{16}}{(1-x^4)(1-x^8)(1-x^{12})} + \cdots}{1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^9}{(1-x^4)(1-x^8)(1-x^{12})} + \cdots}$$

$$= \frac{x}{1+}\,\frac{x}{1+}\,\frac{x^4}{1+}\,\frac{x^3}{1+}\,\frac{x^6}{1+}\cdots$$

(2) I have found empirically that

$$1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^9}{(1-x^4)(1-x^8)(1-x^{12})} + \cdots$$

$$= \frac{(1-x^4)(1-x^6)(1-x^{10})(1-x^{14})\cdots\cdots}{(1-x)(1-x^6)(1-x^{11})(1-x^{16})\cdots\cdots\cdots\cdots(1-x^4)(1-x^9)(1-x^{14})\cdots}$$

and also that

(3) $$x + \frac{x^4}{1-x^4} + \frac{x^9}{(1-x^4)(1-x^8)} + \frac{x^{16}}{(1-x^4)(1-x^8)(1-x^{12})} + \cdots$$

$$= \frac{x(1-x^4)(1-x^6)(1-x^{10})(1-x^{14})\cdots\cdots}{(1-x^2)(1-x^7)(1-x^{12})\cdots\cdots\cdots(1-x^2)(1-x^8)(1-x^{12})\cdots}$$

(4) $$5x\cdot\frac{d}{dx}\left(1 + \frac{x}{1+}\,\frac{x^4}{1+}\,\frac{x^3}{1+}\,\frac{x^6}{1+}\cdots\right)$$

$$= \left\{1 - \frac{(1-x)^5(1-x^4)^5(1-x^3)^5(1-x^6)^5\cdots}{(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})\cdots\cdots}\right\}\cdot\left(1 + \frac{x}{1+}\,\frac{x^4}{1+}\,\frac{x^5}{1+}\cdots\right)$$

(5) If $N = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+}\cdots$

then $$x\left(N + \frac{1}{N}\right)\cdot\frac{(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\cdots}{(1-x)(1-x^4)(1-x^5)\cdots\cdots} = 1 + \sum_{m=1}^{m=\infty}\left(\frac{x^m}{1-x^m} - \frac{x^{5m}}{1-x^{25m}}\right).$$

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(6) If $v = \cfrac{x^{1/5}}{1+}\ \cfrac{x}{1+}\ \cfrac{x^2}{1+}\ \cfrac{x^3}{1+}\ \cfrac{x^4}{1+}\ \cdots$

then $x\left(\frac{1}{v^5}+v^5\right)\cdot\dfrac{(1-x^5)^5(1-x^{10})^5(1-x^{15})^5\cdots}{(1-x)(1-x^2)^5(1-x^3)^5\cdots}$

$= 1 + 6\sum_{n=1}^{n=\infty}\left(\dfrac{n x^n}{1-x^n} - \dfrac{5n x^{5n}}{1-x^{5n}}\right).$

(7) If $2u = 11 + \frac{1}{x}\cdot\left\{\dfrac{(1-x)(1-x^2)(1-x^3)\cdots}{(1-x^5)(1-x^{10})(1-x^{15})\cdots}\right\}^6$

and $2v = 1 + \frac{1}{x^{\frac{1}{5}}}\cdot\left\{\dfrac{(1-x^{\frac{1}{5}})(1-x^{\frac{2}{5}})(1-x^{\frac{3}{5}})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\right\}$

then $\sqrt[5]{\sqrt{u^2+1}-u} = \sqrt{v^2+1} - v$

$= \cfrac{\sqrt[5]{x}}{1+}\ \cfrac{x}{1+}\ \cfrac{x^2}{1+}\ \cfrac{x^3}{1+}\ \cfrac{x^4}{1+}\ \cdots$

(8) If $2u = 11 + 125x\cdot\left\{\dfrac{(1-x^5)(1-x^{10})(1-x^{15})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\right\}^6$

and $2v = 1 + 5x\cdot\left\{\dfrac{(1-x^{25})(1-x^{50})(1-x^{75})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}\right\}$

then, $\dfrac{\sqrt{5}}{1+\left(\frac{\sqrt5-1}{2}\right)\sqrt[5]{\sqrt{u^2+1}-u}} = \dfrac{\sqrt{5}}{1+\left(\frac{\sqrt5-1}{2}\right)\left(\sqrt{v^2+1}-v\right)}$

$= \dfrac{1+\sqrt5}{2} + \cfrac{x}{1+}\ \cfrac{x^5}{1+}\ \cfrac{x^{10}}{1+}\ \cfrac{x^{15}}{1+}\ \cdots$

(9) If $\alpha\beta = \frac{\pi^2}{5}$, then

$$\left\{\left(\frac{\sqrt5+1}{2}\right)^5 + \left(\cfrac{e^{-2\alpha/5}}{1+}\ \cfrac{e^{-2\alpha}}{1+}\ \cfrac{e^{-4\alpha}}{1+}\ \cdots\right)^5\right\}$$

$$\times\left\{\left(\frac{\sqrt5+1}{2}\right)^5 + \left(\cfrac{e^{-2\beta/5}}{1+}\ \cfrac{e^{-2\beta}}{1+}\ \cfrac{e^{-4\beta}}{1+}\ \cdots\right)^5\right\}$$

$$= 5^{-}\sqrt5\left(\frac{\sqrt5+1}{2}\right)^5$$

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(10) If $u = \frac{\sqrt{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\cdots$

and $v = \frac{\sqrt{x^2}}{1+}\frac{x^2}{1+}\frac{x^4}{1+}\frac{x^6}{1+}\cdots$

then

(a). $\frac{v-u^2}{v+u^2} = uv^2$

(b) $UV^2(U^2+V) + U^2 - V + 10UV(UV-U+V+1) = 0$

(c) U and V are satisfied by the relation

when $t \le \sqrt{5}-2$, $U = t\cdot\left(\frac{1-t}{1+t}\right)^2$ and $V = t^2\cdot\frac{1+t}{1-t}$.

(11). If $u = \frac{\sqrt{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\cdots$ and

$v = \frac{\sqrt{x^3}}{1+}\frac{x^3}{1+}\frac{x^6}{1+}\frac{x^9}{1+}\cdots$

then $(v-u^3)(1+uv^3) = 3u^2v^2$.

(12) If $u = \frac{\sqrt{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\cdots$

and $v = \frac{\sqrt{x}}{1-}\frac{x}{1+}\frac{x^2}{1-}\frac{x^3}{1+}\cdots$

then

$uv(u-v)^4 - u^2v^2(u-v)^2 + (1+u^5v^5)(u-v) + 2u^3v^3 = 0$

(13). If $u = \frac{\sqrt{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\cdots$ and $v = \frac{\sqrt{x^4}}{1+}\frac{x^5}{1+}\frac{x^8}{1+}\cdots$

then,

$u^5v^5 + uv + (u^5+v^5)(uv-1) = 5u^2v^2(uv-1)^2$

(14) If $u = \frac{\sqrt{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\cdots$ and $v = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+}\cdots$

then $u^5 = v\cdot\frac{1-2v+4v^2-3v^3+v^4}{1+3v+4v^2+2v^3+v^4}$

(15) I have also found empirically the following results:

If $v = \cfrac{\sqrt[3]{x}}{1+\cfrac{x+x^2}{1+\cfrac{x^2+x^4}{1+\cfrac{x^3+x^6}{1+\cfrac{x^4+x^8}{1+\cdots}}}}}$

then

(a) $v = \sqrt[3]{x}\cdot\dfrac{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots}{(1-x^3)^3(1-x^9)^3(1-x^{15})^3(1-x^{21})^3\cdots}$

(b) $\dfrac{1}{v} = \dfrac{1}{\sqrt[3]{x}}\cdot\dfrac{1+\sqrt[3]{x}+\sqrt[3]{x^3}+\sqrt[3]{x^6}+\sqrt[3]{x^{10}}+\cdots}{1+x+x^3+x^6+x^{10}+\cdots} - 1$

$= \sqrt[3]{\left\{\dfrac{1}{x}\cdot\left(\dfrac{1+x+x^3+x^6+x^{10}+\cdots}{1+x^3+x^9+x^{18}+x^{30}+\cdots}\right)^4 - 1\right\}}$

(c) $2v = 1 - \dfrac{1-2\sqrt[3]{x}+2\sqrt[3]{x^4}-2\sqrt[3]{x^9}+2\sqrt[3]{x^{16}}-\cdots}{1-2x+2x^4-2x^9+2x^{16}-\cdots}$

$= \sqrt[3]{\left\{1-\left(\dfrac{1-2x+2x^4-2x^9+2x^{16}-\cdots}{1-2x^3+2x^{12}-2x^{27}+2x^{48}-\cdots}\right)^4\right\}}$

(d) $\dfrac{1}{v}+4v^2 = 3 + \dfrac{1}{x^{\frac{1}{3}}}\cdot\left\{\dfrac{(1-x^{\frac{1}{3}})(1-x^{\frac{2}{3}})(1-x^{\frac{3}{3}})(1-x^{\frac{4}{3}})\cdots}{(1-x^3)(1-x^6)(1-x^9)(1-x^{12})\cdots}\right\}^8$

$= \sqrt[3]{\left[27 + \dfrac{1}{x}\cdot\left\{\dfrac{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}{(1-x^3)(1-x^6)(1-x^9)(1-x^{12})\cdots}\right\}^{12}\right]}$

and many results analogous to the previous. continued fraction.

$$\cfrac{e^{-\frac{\pi}{5}\sqrt{10}}}{1+\cfrac{e^{-\pi\sqrt{10}}+e^{-2\pi\sqrt{10}}}{1+\cfrac{e^{-2\pi\sqrt{10}}+e^{-4\pi\sqrt{10}}}{1+\cfrac{e^{-3\pi\sqrt{10}}+e^{-6\pi\sqrt{10}}}{1+\&c}}}} = \frac{\sqrt{9+3\sqrt{6}}-\sqrt{7+3\sqrt{6}}}{(1+\sqrt{5})\sqrt{\sqrt{5}+\sqrt{6}}}.$$

(102)

$$1+240\left(\frac{1^3q}{1-q}+\frac{2^3q^2}{1-q^2}+\cdots\right)=\left(\frac{2K}{\pi}\right)^4(1+14k^2+k^4)$$

$$1+240\left(\frac{1^3q'}{1-q'}+\frac{2^3q'^2}{1-q'^2}+\cdots\right)=\left(\frac{K'}{\pi}\right)^4(1+14k'^2+k'^4)$$

$$1+240\left(\frac{1^3q^2}{1-q^2}+\frac{2^3q^4}{1-q^4}+\cdots\right)=\left(\frac{2K}{\pi}\right)^4\{1-(kk')^2\}$$

$$1-504\left(\frac{1^5q}{1-q}+\frac{2^5q^2}{1-q^2}+\frac{3^5q^3}{1-q^3}+\cdots\right)=\left(\frac{2K}{\pi}\right)^6(1+k^2)(1-24k^2+k^4)$$

$$1-504\left(\frac{1^5q'}{1-q'}+\frac{2^5q'^2}{1-q'^2}+\frac{3^5q'^3}{1-q'^3}+\cdots\right)=\left(\frac{K'}{\pi}\right)^6(1+k'^2)(1-24k'^2+k'^4)$$

$$1-504\left(\frac{1^5q^2}{1-q^2}+\frac{2^5q^4}{1-q^4}+\cdots\right)=\left(\frac{2K}{\pi}\right)^6(k'^2-k^2)\left\{1+\tfrac{1}{2}(kk')^2\right\}$$

$$(2^s-1)\frac{B_s}{2s}+\frac{1^{s-1}q}{1-q}+\frac{2^{s-1}q^2}{1+q^2}+\frac{3^{s-1}q^3}{1-q^3}+\cdots$$

$$=2^s\left\{\frac{B_s}{2s}-\left(\frac{1^{s-1}q^2}{1-q^2}+\frac{2^{s-1}q^4}{1-q^4}+\frac{3^{s-1}q^6}{1-q^6}+\cdots\right)\right\}$$

$$-\left\{\frac{B_s}{2s}-\left(\frac{1^{s-1}q}{1-q}+\frac{2^{s-1}q^2}{1-q^2}+\frac{3^{s-1}q^3}{1-q^3}+\cdots\right)\right\}.$$

$$(2^s-1)\frac{B_s}{2s}+\frac{1^{s-1}q}{1+q}+\frac{2^{s-1}q^2}{1-q^2}+\frac{3^{s-1}q^3}{1+q^3}+\cdots$$

$$=2^s\left\{\frac{B_s}{2s}+\left(\frac{1^{s-1}q^2}{1-q^2}+\frac{2^{s-1}q^4}{1-q^4}+\frac{3^{s-1}q^6}{1-q^6}+\cdots\right)\right\}$$

$$+\left\{\frac{B_s}{2s}+\left(\frac{1^{s-1}q}{1-q}+\frac{2^{s-1}q^2}{1-q^2}+\frac{3^{s-1}q^3}{1-q^3}+\cdots\right)\right\}$$

$$-2\left\{\frac{B_s}{2s}+\left(\frac{1^{s-1}q^2}{1-q^2}+\frac{2^{s-1}q^4}{1-q^4}+\frac{3^{s-1}q^6}{1-q^6}+\cdots\right)\right\}$$

Add to 18(109) (109)

~~The constant in the series $\sum \frac{d(n)}{\sqrt{n}}$ is A.~~

Let $d'(n) = d(n) - 2\gamma - \log n$ and ~~let the constant be~~

$$\frac{2\gamma+\log 1}{\sqrt{1}} + \frac{2\gamma+\log 2}{\sqrt{2}} + \cdots + \frac{2\gamma+\log n}{\sqrt{n}}$$
$$= C + 2\sqrt{n}(\log n + 2\gamma - 2) + o(1).$$

Consider now the Voronoi identity for the remainder function of any convex function

Then
$$\sum_{n=1}^{n=\infty} \frac{d'(n)}{\sqrt{n}} = \left|\zeta\left(\tfrac{1}{2}\right)\right|^2 - C.$$

TO

$$\sum_{2\sqrt{n}<\omega} d(n)\left(1 - \frac{\omega}{\pi\sqrt{n}}\right) + \sum_{2\sqrt{n}>\omega} d(n)\left\{\frac{2}{\pi}\sin^{-1}\frac{\omega}{2\sqrt{n}} - \frac{\omega}{\pi\sqrt{n}}\right\}$$
$$= \sum_{2\sqrt{n}<\omega} d(n) - \sum_{2\sqrt{n}<\omega} d'(n)\frac{\omega}{\pi\sqrt{n}} + \sum_{2\sqrt{n}>\omega} d'(n)\left\{\frac{2}{\pi}\sin^{-1}\frac{\pi}{2\sqrt{n}} - \frac{\omega}{\pi\sqrt{n}}\right\}$$
$$- \sum_{2\sqrt{n}<\omega} \frac{(2\gamma+\log n)\omega}{\pi\sqrt{n}} + \sum_{2\sqrt{n}>\omega} (2\gamma+\log n)\left\{\frac{2}{\pi}\sin^{-1}\frac{\pi}{2\sqrt{n}} - \frac{\omega}{\pi\sqrt{n}}\right\}$$
$$= \sum_{2\sqrt{n}<\omega} d(n) - \frac{\omega}{\pi}\left\{\left|\zeta\left(\tfrac{1}{2}\right)\right|^2 - C\right\} + \sum_{2\sqrt{n}>\omega} d'(n)\cdot\frac{2}{\pi}\sin^{-1}\frac{\pi}{2\sqrt{n}}$$
$$- \frac{\omega}{\pi}\left\{C + 2\sqrt{n}(\log n - \right. - \frac{\omega}{\pi}\sum_{2\sqrt{n}<\omega}\frac{2\gamma+\log n}{\sqrt{n}}$$
$$\left. + \sum_{2\sqrt{n}>\omega}(2\gamma+\log n)\left\{\frac{2}{\pi}\sin^{-1}\frac{\pi}{2\sqrt{n}} - \frac{\omega}{\pi\sqrt{n}}\right\}\right.$$

$$\sum_{2\sqrt{n}<\omega} d(n) + \frac{2}{\pi}\sum_{2\sqrt{n}>\omega} d'(n)\sin^{-1}\frac{\pi}{2\sqrt{n}} + \frac{\omega}{\pi}\left(C - \sum_{2\sqrt{n}<\omega}\frac{2\gamma+\log n}{\sqrt{n}}\right)$$
$$+ \sum_{2\sqrt{n}>\omega}(2\gamma+\log n)\left\{\frac{2}{\pi}\sin^{-1}\frac{\pi}{2\sqrt{n}} - \frac{\omega}{\pi\sqrt{n}}\right\}$$
$$= \frac{1}{2}\omega^2 + \frac{1}{2}\omega - \frac{1}{2}\sum_{q=1}^{q=\infty}\frac{d(q)}{q}e^{-2\pi\omega\sqrt{q}} + \frac{1}{2}\sum_{q=1}^{q=\infty}\frac{d(q)}{q}\cos 2\pi\omega.$$

Idea You can replace the Bessel fn of the V. identity by circular functions, at the price of complicating the sum, interesting idea, but lacks of use value for the study of the divisor problem

dont know [illegible] (110)
Add me a 94[18(100)]

I shall conclude this paper by considering two functions which can be expressed as polynomials in P, Q and R, viz

$$U_m = \frac{1^{m+1} - 3^{m+1}x + 5^{m+1}x^3 - 7^{m+1}x^6 + 9^{m+1}x^{10} - \cdots}{1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \cdots},$$

$$V_m = \frac{1^m - 5^m x - 7^m x^2 + 11^m x^5 + 13^m x^7 - \cdots}{1 - x - x^2 + x^5 + x^7 - \cdots},$$

where m is a positive even integer. It is easy to show that

$$U_{m+2} = PU_m + 8x\frac{dU_m}{dx}, \quad V_{m+2} = PV_m + 24x\frac{dV_m}{dx},$$

using the relations

$$Q^3 - R^2 = 1728x(1 - 3x + 5x^3 - 7x^6 + \cdots)^8$$
$$= 1728x(1 - x - x^2 + x^5 + x^7 - \cdots)^{24}.$$

It is evident that

$$U_0 = V_0 = 1.$$

From ( ) and ( ) we obtain the following results.

Table

1. $U_2 = P.$ $\quad V_2 = P.$
2. $U_4 = \frac{1}{8}(5P^2 - 2Q).$ $\quad V_4 = 3P^2 - 2Q.$
3. $U_6 = \frac{1}{9}(35P^3 - 42PQ + 16R).$ $\quad V_6 = 15P^3 - 30PQ + 16R.$
4. $\begin{cases} U_8 = \frac{1}{9}(35P^4 - 84P^2Q + 64PR - 12Q^3). \\ V_8 = 105P^4 - 420P^2Q + 448PR - 132Q^2. \end{cases}$
5. $\begin{cases} U_{10} = \frac{1}{9}(385P^5 - 1540P^3Q + 1760P^2R - 660PQ^2 + 64QR). \\ V_{10} = 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR. \end{cases}$

In general U_{2S} and V_{2S} are of the form

$$\sum K_{\ell,m,n} P^\ell Q^m R^n,$$

where $\quad \ell + 2m + 3n = S.$

(11)

(1) $\frac{9}{5} + \sqrt{\frac{8}{5}} = 3.14164\ldots = \pi + .00005$

(2) $\frac{4}{\pi} = 1 + \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \ldots$

(3) $\frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{88}{64^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{131}{64^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \ldots$

(4) $\frac{8(1+\sqrt{2})}{\pi} = (6+\sqrt{2}) + (66+19\sqrt{2})\left(\frac{1}{2}\right)^3 \cdot \frac{1}{64}\left(\frac{\sqrt{2}-1}{2}\right)^8 +$

$(126+37\sqrt{2})\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 \cdot \frac{1}{64^2}\left(\frac{\sqrt{2}-1}{2}\right)^{16} + \ldots$

$$\prod_{n=1}^{n=\infty}\left\{\frac{(1+\frac{a}{m}h^{n-1})(1+\frac{b}{m}h^{n-1})}{(1-ah^{n-1})(1-bh^{n-1})}\cdot\frac{1-h^n}{1+mh^n}\cdot\frac{1-k^n}{1+\frac{k^n}{m}}\right\}$$

$$= 1 + (1+m)\sum_{n=1}^{n=\infty}\left(\frac{a^n}{m+h^n} + \frac{b^n}{1+mk^n}\right)$$

when $k = ab$

$$\frac{\pi m K}{2}\left\{\frac{1}{(\pi K)^2} + 4\sum_{n=1}^{n=\infty}\frac{q^n}{1+q^{2n}}\cdot\frac{1}{(mK)^2+(\pi n)^2}\right\}$$

$$= \frac{1}{m} + \frac{(h)^n}{m} + \frac{a^n}{m} + \frac{(ah)^n}{m} + \frac{b^n}{m} + \frac{(b-h)^n}{m} + \ldots$$

not proved

If $\frac{\pi\alpha}{2} = gd\left(\frac{\pi\beta}{2}\right)$, then

$$\left(\frac{1^n-\alpha^n}{1^n+\beta^n}\right)\cdot\left(\frac{3^n+\beta^n}{3^n-\alpha^n}\right)^3\cdot\left(\frac{5^n-\alpha^n}{5^n+\beta^n}\right)^5\cdot\left(\frac{7^n+\beta^n}{7^n-\alpha^n}\right)^7\ldots\ldots\ldots = e^{-\frac{\pi\alpha\beta}{2}}.$$

$$\left\{(1+x)e^{-x}\right\}\left\{\left(1+\frac{x}{2}\right)^4 e^{-x}\right\}\left\{\left(1+\frac{x}{3}\right)^9 e^{-x}\right\}\ldots\ldots\ldots = e^{\frac{\pi}{2}}$$

if $x = \left\{\frac{\log\left(\frac{1+\sqrt{5}}{2}\right)}{\pi}\right\}^2$

$$\frac{(1+ax)(1+ax^2)(1+ax^3)\ldots\ldots}{(1-bx)(1-bx^2)(1-bx^3)\ldots\ldots}$$

$$= 1 + \frac{(a+b)x}{(1-x)(1-bx)} + \frac{(a+b)(a+bx)x^3}{(1-x)(1-x^2)(1-bx)(1-bx^2)} +$$

$$\frac{(a+b)(a+bx)(a+bx^2)x^6}{(1-x)(1-x^2)(1-x^3)(1-bx)(1-bx^2)(1-bx^3)} + \ldots\ldots$$

Please mention in a foot note above the result (1.61) that there was a mistake in Comptes Rendus concerning the value of $p(61)$ over $p(1)$. I think it was given there as -44.641 instead of the correct value -74.641. Please ascertain if there was really a mistake; the figures might not have been mentioned in Comptes Rendus. In yesterday's letter instead of writing $\frac{\pi}{\sin \pi \alpha} \sin \pi \alpha$ I inverted them in haste. Is it not better to construct a small table showing the maximum value $M_5(\alpha)$ and minimum $m_5(\alpha)$ of $|C_5(\alpha)|$ in the fourth strip parallel like this

α	1	2	3	4	.....	15
$M_5(\alpha)$	1	1	$2\cos\frac{\pi}{15}$	$2\cos\frac{\pi}{15}$	.....	.
$m_5(\alpha)$	1	1	$2.5\sin\frac{\pi}{15}$	$2.5\sin\frac{\pi}{15}$	.....	

$$x\log 1 + x^2\log 2 + x^3\log 3 + x^4\log 4 + x^5\log 5 + \&c \quad \ldots\ldots (1)$$

$$= \log 2\,(x^2 + 2x^4 + x^6 + 3x^8 + x^{10} + 2x^{12} + x^{14} + 4x^{16} + \&c)$$

$$+ \log 3\,(x^3 + x^6 + 2x^9 + x^{12} + x^{15} + 2x^{18} + x^{21} + x^{24} + \&c)$$

$$+ \log 5\,(x^5 + x^{10} + x^{15} + x^{20} + 2x^{25} + x^{30} + x^{35} + \&c)$$

$$+ \log 7\,(x^7 + x^{14} + x^{21} + x^{28} + x^{35} + x^{42} + 2x^{49} + \&c)$$

$$+ \&c \qquad \&c \qquad \&c \qquad \&c$$

where 2, 3, 5, 7, 11 &c are prime numbers.

$$= \log 2\left(\frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} + \frac{x^8}{1-x^8} + \frac{x^{16}}{1-x^{16}} + \&c\right)$$

$$+ \log 3\left(\frac{x^3}{1-x^3} + \frac{x^9}{1-x^9} + \frac{x^{27}}{1-x^{27}} + \frac{x^{81}}{1-x^{81}} + \&c\right)$$

$$+ \log 5\left(\frac{x^5}{1-x^5} + \frac{x^{25}}{1-x^{25}} + \frac{x^{125}}{1-x^{125}} + \&c\right)$$

$$+ \log 7\left(\frac{x^7}{1-x^7} + \frac{x^{49}}{1-x^{49}} + \frac{x^{343}}{1-x^{343}} + \&c\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

changing x to $-e^{-x}$ in (1) and (2) we have.

$$\log 3\left(\frac{1}{e^{3x}+1} + \frac{1}{e^{9x}+1} + \frac{1}{e^{27x}+1} + \&c\right)$$

$$+ \log 5\left(\frac{1}{e^{5x}+1} + \frac{1}{e^{25x}+1} + \frac{1}{e^{125x}+1} + \&c\right)$$

$$+ \log 7\left(\frac{1}{e^{7x}+1} + \frac{1}{e^{49x}+1} + \frac{1}{e^{343x}+1} + \&c\right)$$

$$+ \ldots\ldots\ldots \quad 3, 5, 7 \text{ \&c being prime numbers}$$

$$= \log 2\left(\frac{1}{e^{2x}-1} + \frac{1}{e^{4x}-1} + \frac{1}{e^{8x}-1} + \frac{1}{e^{16x}-1} + \&c\right)$$

$$+ e^{-x}\log 1 - e^{-2x}\log 2 + e^{-3x}\log 3 - \&c \quad - \quad (3).$$

Let us suppose,

$$\phi(x) + \log 2\,(2e^{-2x} + 4e^{-4x} + 8e^{-8x} + 16e^{-16x} + \&c)$$

$$= \log 2\,(e^{-2x} + e^{-4x} + e^{-8x} + e^{-16x} + \&c)$$

$$+ \log 3\,(e^{-3x} + e^{-9x} + e^{-27x} + e^{-81x} + \&c)$$

$$+ \log 5\,(e^{-5x} + e^{-25x} + e^{-125x} + e^{-625x} + \&c)$$

$$+ \ldots\ldots\ldots$$ 2, 3, 5 &c being prime nos

Then we see that,

$$\phi(x) - \phi(2x) + \phi(3x) - \phi(4x) + \phi(5x) - \&c$$

$$= \log 2\,\left(\frac{1}{e^{2x}+1} + \frac{1}{e^{4x}+1} + \frac{1}{e^{8x}+1} + \&c\right)$$

$$+ \log 3\,\left(\frac{1}{e^{3x}+1} + \frac{1}{e^{9x}+1} + \frac{1}{e^{27x}+1} + \&c\right)$$

$$+ \log 5\,\left(\frac{1}{e^{5x}+1} + \frac{1}{e^{25x}+1} + \frac{1}{e^{125x}+1} + \&c\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \log 2\,\left(\frac{2}{e^{2x}+1} + \frac{4}{e^{4x}+1} + \frac{8}{e^{8x}+1} + \&c\right)$$

But we know that

$$\frac{1}{1-e^{-2x}} = (1+e^{-2x})(1+e^{-4x})(1+e^{-8x})(1+e^{-16x})\,\&c$$

Taking logarithmic differentiations, we have

$$\frac{2}{e^{2x}-1} = \frac{2}{e^{2x}+1} + \frac{4}{e^{4x}+1} + \frac{8}{e^{8x}+1} + \&c \qquad (A)$$

Also we know that $\frac{1}{e^{2x}+1} = \frac{1}{e^{2x}-1} - \frac{2}{e^{4x}-1}$

Hence, $\frac{1}{e^{2x}+1} + \frac{1}{e^{4x}+1} + \frac{1}{e^{8x}+1} + \ldots\ldots$

$$= \left(\frac{1}{e^{2x}-1} - \frac{2}{e^{4x}-1}\right) + \left(\frac{1}{e^{4x}-1} - \frac{2}{e^{8x}-1}\right) + \ldots\ldots$$

$$= \frac{1}{e^{2x}-1} - \frac{1}{e^{4x}-1} - \frac{1}{e^{8x}-1} - \frac{1}{e^{16x}-1} - \&c \qquad (B)$$

Therefore, by substituting (6) and (7) in (5) we see

$$\phi(x) - \phi(2x) + \phi(3x) - \phi(6x) + \&c$$

$$= \log 2\left(\frac{1}{e^{x}-1} - \frac{1}{e^{2x}-1} - \frac{1}{e^{3x}-1} - \&c\right)$$

$$+ \log 3\left(\frac{1}{e^{3x}+1} + \frac{1}{e^{9x}+1} + \frac{1}{e^{27x}+1} + \&c\right)$$

$$+ \log 5\left(\frac{1}{e^{5x}+1} + \frac{1}{e^{25x}+1} + \frac{1}{e^{125x}+1} + \&c\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \frac{2\log 2}{e^{2x}-1}.$$

$$= e^{-x}\log 1 - e^{-2x}\log 2 + e^{-3x}\log 3 - \&c. \text{ by (3).}$$

= a finite quantity when x approaches 0.

Hence $\phi(x)$ is finite when x approaches 0. — (8)

Applying (8) in (4) we see that,

$$\log 2\left(e^{-2x} + e^{-4x} + e^{-8x} + \ldots\ldots\right)$$

$$+ \log 3\left(e^{-3x} + e^{-9x} + e^{-27x} + \ldots\ldots\right)$$

$$+ \log 5\left(e^{-5x} + e^{-25x} + e^{-125x} + \ldots\ldots\right)$$

$$+ \log 7\left(e^{-7x} + e^{-49x} + e^{-343x} + \ldots\ldots\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$- \log 2\left(2e^{-2x} + 4e^{-4x} + 8e^{-8x} + 16e^{-16x} + \ldots\right)$$

is finite when x approaches 0 ——— (9)

Now let us suppose that Z is a function of n such that there are n prime numbers within Z. Then from (9) we see that:

$$\int^{\infty} \log z \left(e^{-zx} + e^{-z^2x} + e^{-z^3x} + e^{-z^4x} + \&c\right) dz$$

$$- \log 2 \int^{\infty} 2^y e^{-2^y x} dy \text{ is finite when } x \text{ becomes}$$

that is,

$$\int^{\infty} \log z \left(e^{-zx} + e^{-z^2x} + e^{-z^3x} + \ldots\right) dz,$$

$$- \frac{1}{x} \text{ is finite when } x \text{ becomes } 0. \quad \text{---(10)}$$

Again supposing $\frac{dn}{dz} = \frac{\psi(z)}{\log z}$ and substituting this in (10) we see that $\psi(z)$ is such a function that,

$$\int^{\infty} \left(e^{-zx} + e^{-z^2x} + e^{-z^3x} + \ldots\right) \psi(z)\, dz$$

$$- \frac{1}{x} \text{ is finite when } x \text{ becomes } 0. \quad \text{---(11)}$$

Hence from (11) we see that

$$\psi(z) = 1 - \frac{1}{2\sqrt{z}} - \frac{1}{3 z^{\frac{1}{3}}} - \frac{1}{5.2 z^{\frac{1}{5}}} + \&c$$

Therefore $\frac{dn}{dz} = \frac{z - \frac{\sqrt{z}}{2} - \frac{\sqrt[3]{z}}{3} - \&c}{z \log z}$ ——— (12)

Hence the no of prime numbers within z

$$= \int^{z} \frac{u - \frac{\sqrt{u}}{2} - \frac{\sqrt[3]{u}}{3} - \frac{\sqrt[5]{u}}{5} + \&c}{u \log u} du$$

$$= \int^{z} \frac{du}{\log u} - \frac{1}{2}\int^{\sqrt{z}} \frac{du}{\log u} - \frac{1}{3}\int^{\sqrt[3]{z}} \frac{du}{\log u} + \&c$$

$$- \frac{1}{5}\int^{\sqrt[5]{z}} \frac{du}{\log u} + \&c \quad \text{———} \quad (13)$$

For all values of x I have proved that-

1. $\log 2 \left\{ e^{-x} + 2e^{-2x} + 4e^{-4x} + 8e^{-8x} + 16e^{-16x} + \&c \right.$
$\left. + 1 - \frac{x}{3\,\underline{|1}} + \frac{x^2}{7\,\underline{|2}} - \frac{x^3}{15\,\underline{|3}} + \frac{x^5}{31\,\underline{|5}} - \frac{x^{10}}{63\,\underline{|5}} + \&c \right\}$
$= \frac{1}{x} + F(x)$, where $x F(x)$ correct to 10 places of decimals $= .000009884 \cos\left(\frac{8\pi^2 \log x}{\log 2} + .872811\right)$,

So my assertions in the previous letter that-
"$\log 2 \left(2e^{-2x} + 4e^{-4x} + 8e^{-8x} + \&c\right) - \int_0^\infty 2^y e^{-x.2^y} dy$
is finite when $x = 0$" is wrong.

2. The previous above $F(x)$ has also the property viz. $F(x) - F(2x) + F(3x) - F(4x) + \&c = 0$
Hence the assertion that,
"If $\phi(x) - \phi(2x) + \phi(3x) - \phi(4x) + \&c$ be finite when $x = 0$, then $\phi(x)$ is necessarily finite when $x = 0$"
is also wrong, as you remarked.
But one thing is certain; that $\phi(x)$ when $x = 0$ either cannot be determinately infinite but will be finite or indeterminate, if infinitely often going to infinity as well as vanishing like the function
$\frac{a \cos(b + c \log x)}{x^m}$.
So a matter of fact $\phi(x)$ is not finite when $x = 0$ in the present case.

3. I think I am correct in using $\frac{dn}{dz}$ which is not the differential coeff^{t} of not a discontinuous function but the diff^{l} coeff^{t} of ~~a regular~~ an average continuous function passing fairly (though not exactly) through the isolated points. I have used $\frac{dn}{dz}$ in finding the number of nos of the form $2^p \cdot 3^q$, $2^p + 3^q$ &c less than Z. and have got correct results.

4. $\int_0^\infty \left(e^{-2^2 x} + e^{-2^4 x} + e^{-2^8 x} + \&c\right) \log z \, dx = \frac{1}{x} + f(x)$.
when x approaches 0, where $f(x)$ is of lower

order than $\frac{1}{2}$ and indeterminate when $x=0$. (4)
which can be found if we know the value of

$\Sigma \log p \ (e^{-px} + e^{-p^2x} + e^{-p^3x} + \&c) - \frac{1}{2x}$ when x is very small, where $p = 2, 3, 5, 7, 11$ &c

5. Even though $Li(x) - \frac{1}{2}Li(\sqrt{x}) - \frac{1}{3}Li(\sqrt[3]{x}) - \frac{1}{5}Li(\sqrt[5]{x})$ + &c does not actually coincide with the no of primes less than n, yet it is the average value of the no. of primes less than n, and so we can use it in finding the sum of a prime nos. and in other similar questions.

Thus in the case of no of diviseurs of n which I have found to be of order $e^{c \cdot \frac{\log n}{\log\log n}}$ where c is some definite and finite constant, the average value is only $2\gamma + \log n$ where γ is the Eulerian Constant.

Also we have, if $a_1, a_2, a_3 \ldots a_n$ are n irregular nos. Δ, then;

$a_1 + a_2 + a_3 + \ldots a_n$
$= \int (\text{average value of } a_n)\, dn + \frac{1}{2} a_n + \theta(n)$.

where $\theta(n)$ is of lower order than the difference between the actual order of a_n and the average order of a_n. Thus we see that

6. $2 + 3 + 5 + 7 + \ldots + z$ (where z is the nth prime number)
$= \left\{ Li(z^2) - \frac{1}{2}Li(z^{\frac{3}{2}}) - \frac{1}{3}Li(z^{\frac{4}{3}}) - \frac{1}{5}Li(z^{\frac{6}{5}}) + \&c \right.$
$\left. + \frac{z}{2} + \theta(z) \right\}$* where $\theta(z)$ is of lower order than $z(\log z)^{-c}$ for any value of c, for which its actual order is less than (z — average value of z)

N.B. The left side i.e the sum of the first n prime numbers is not equal to the right side as it stands but will be equal to the right side transformed to a function of n assuming that $n = Li(z) - \frac{1}{2}Li(\sqrt{z}) - \frac{1}{3}Li(\sqrt[3]{z}) - \&c$
The above is not clearly written and I shall clearly explain to you afterwards what I mean.

yours very sincerely
S. Ramanujan

92 (1)

Let

$$S_{\nu,h} = h \sum_{\nu=-\infty}^{\nu=\infty} \phi(\nu + \nu h),$$

where ν is any number, $h > 0$ and $\phi(z)$ is an entigral function such that

$$\phi(z)\, z^{1+\varepsilon}\, e^{2i\pi m z} \longrightarrow 0$$

as $z \to \infty$ in the upper half of the z plane and

$$\phi(z)\, z^{1+\varepsilon}\, e^{-2i\pi n z} \longrightarrow 0$$

in the lower half of the plane where $\varepsilon > 0$, $m \geq 0$ and $n \geq 0$. Then

Theorem I

$$S_{\nu,h} = A_1 S_{\nu_1,h} + A_2 S_{\nu_2,h} + A_3 S_{\nu_3,h} + \cdots$$

where $h > 0$, $\nu, \nu_1, \nu_2, \ldots$ are any numbers and $A_1, A_2, A_3, \ldots$ satisfy the simultaneous equations

$$A_1 e^{\frac{2i\pi\nu_1}{h}[mh]} + A_2 e^{\frac{2i\pi\nu_2}{h}[mh]} + \cdots \text{ to } 1 + [mh] + [nh] \text{ terms} = e^{\frac{2i\pi\nu}{h}[mh]}$$

$$A_1 e^{\frac{2i\pi\nu_1}{h}([mh]-1)} + A_2 e^{\frac{2i\pi\nu_2}{h}([mh]-1)} + \cdots \text{ to } 1 + [mh] + [nh] \text{ terms} = e^{\frac{2i\pi\nu}{h}([mh]-1)}$$

$$A_1 e^{\frac{2i\pi\nu_1}{h}([mh]-2)} + A_2 e^{\frac{2i\pi\nu_2}{h}([mh]-2)} + \cdots \text{ to } 1 + [mh] + [nh] \text{ terms} = e^{\frac{2i\pi\nu}{h}([mh]-2)}$$

..... to $1 + [mh] + [nh]$ equations

Theorem II